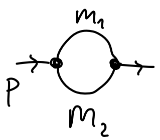
THE GEOMETRY OF SCATTERING AMPLITUDES

(j.w. Erik Panzer)

1. INTRODUCTION
2. SETUP / MAIN EXAMPLE
3. LANDAU VARIETIES
4. PICARD-LEFSCHETZ THEORY
5. ITERATED VARIATIONS/DISCONTINUITIES

Scattering amplitudes, e.g.

I^D 

(p^2, m_1^2, m_2^2)

$$= \int_{\mathbb{R}^{1,D-1}} d^D k \; \frac{1}{k^2 - m_1^2} \cdot \frac{1}{(k+p)^2 - m_2^2}$$

"momentum space representation"

$$= \int_{\{[x_1 : x_2] \,|\, x_i > 0\} \subset \mathbb{P}^1(\mathbb{C})} \frac{x_1 \, dx_2 - x_2 \, dx_1}{\mathcal{U}^{D-2} \cdot \mathcal{F}^{2 - D/2}}$$

"parametric represent."

where

$$\mathcal{U} = x_1 + x_2$$

$$\mathcal{F} = (x_1 + x_2) \cdot (m_1^2 x_1 + m_2^2 x_2) - p^2 x_1 x_2$$

defines meromorphic (multi-valued) function on a subset $T' \subset T = \{$ complex kinematics : masses, momenta, ... $\}$

Q:

1. Determine T' ?

2. Analytic structure of $I : T' \to \mathbb{C}$?

(important for physics: branch points $\leftrightarrow$ thresholds for real particle generation etc...)

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For $\left\{ \begin{array}{l} X, T \text{ (compact) complex manifolds }, \quad n = \dim_{\mathbb{C}} X \\ D = A \cup B \text{ normal crossing / transverse arrangement} \\ \text{of smooth complex hypersurfaces in } X \times T \\ \left(A = \bigcup_{i \in I} A_i \,,\ B = \bigcup_{j \in J} B_j \ ;\ A_t = A \cap X \times \{t\} \,,\ B_t = \dots \right) \end{array} \right.$

consider $$I(t) = \int_{\sigma_t} \omega_t$$

with $\left\{ \begin{array}{l} \omega_t \in \Omega^n(X \setminus A_t, B_t) \text{ closed (analytic) } n\text{-form} \\ \sigma_t \in C_n(X \setminus A_t, B_t) \text{ relative } n\text{-cycle} \end{array} \right.$

Example

$$I(t) = \log(1-t) = \int_{\sigma_t} \omega_t$$

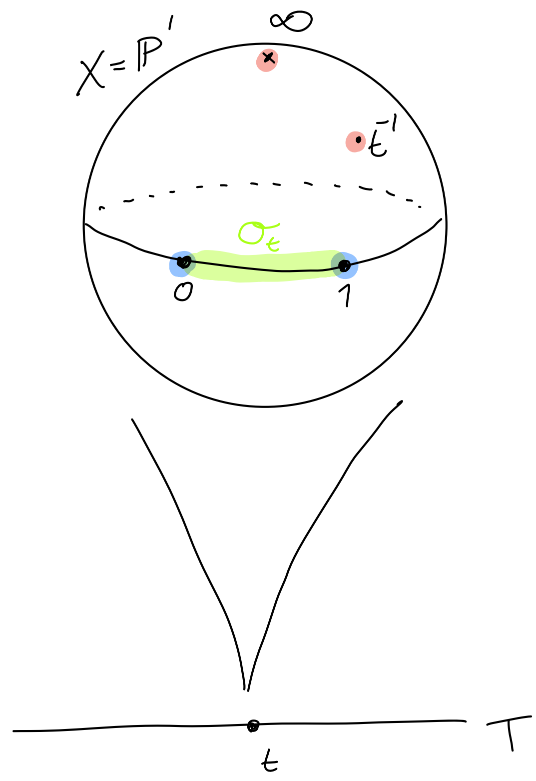
$$\sigma_t = [0,1] \in C_1(X \setminus A_t, B_t)$$

$$\omega_t = \frac{t\,dx}{tx-1} \in \Omega^1(X \setminus A_t, B_t)$$

$$X = T = \mathbb{P}^1$$

$$A = \{tx = 1\} \cup \{x = \infty\}$$

$$B = \{x = 0\} \cup \{x = 1\}$$



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Thm (Pham)

a) If $\exists\, t_0 \in T$ s.t. $I(t_0)$ converges, then I is analytic in a neigborhood of t_0.

b) I can be analytically continued to every $t \in T \setminus L$, where L is the Landau variety (of I).

(this answers Q1 !)

Proof: Pham "Singularities of Integrals"

$D = A \cup B$ normal crossing / transverse

$\Longrightarrow$ $\exists$ canonical stratification

with strata $S_{KL} = \left(\bigcap_{k \in K} A_k\right) \cap \left(\bigcap_{\ell \in L} B_\ell\right)$

(smooth manifolds whose union is $A \cup B$)

Let $\pi : X \times T \to T$, $(x,t) \mapsto t$.

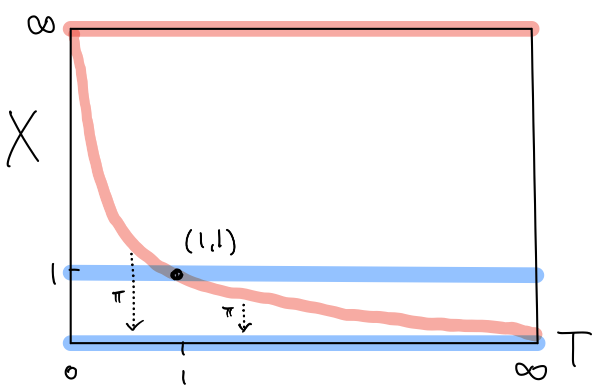
<u>Thm</u> (Thom) If $\pi|_S$ is a submersion for each stratum S, then π is a <u>stratified fiber bundle</u>,

i.e., $\exists$ local trivializations that trivialize each $\pi|_S$ simultaneously.

Def The <u>Landau variety</u> $L = L(\pi, \dots)$ is the codimension one subset of $\{ t \in T \mid t \text{ crit. value of some } \pi|_S \}$.

(Riemann's extension theorem ...)

Example ($I(t) = \log(1-t)$)



$\Rightarrow \quad L = \{0, 1, \infty\}$

(no singularity of I !)

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To study the singularities of I we consider the <u>monodromy representation</u>

$$\rho : \pi_1(T \setminus L, t_0) \to \operatorname{Aut} H_n(X \setminus A_{t_0}, B_{t_0})$$

or the <u>variation operators</u>

$$\begin{aligned} \mathrm{Var}_\gamma : H_n(X \setminus A_{t_0}, B_{t_0}) &\to H_n(X \setminus A_{t_0}, B_{t_0}) \\ h &\mapsto \rho(\gamma).h - h \, . \end{aligned}$$

Assume $T = \mathbb{P}^1$ and $L = \{0\}$, $p \in \pi^{-1}(0)$ critical point of $\pi|_{S_{KL}}$ $\left(S_{KL} = \left(\bigcap_{k\in K} A_k\right) \cap \left(\bigcap_{l\in L} B_l\right) \right)$

(plus some technical conditions, "simple pinch point"
think isolated ...)

Milnor: $\exists\, W$ neighborhood of p containing no other critical points. Can shrink W such that $W \cap A = \bigcup_{k\in K} A_k$, $W \cap B = \bigcup_{l\in L} B_l$.

<u>Def / Prop</u> The groups $H_n(W, A_t \cup B_t)$ and $H_n(W \setminus A_t, B_t)$ are both one-dimensional, generated by the

- <u>vanishing cell</u> $\tau \in C_n(W, A_t \cup B_t)$, (defined in local coordinates, vanishes as $t \to 0$)

- <u>vanishing cycle</u> $\nu \in C_n(W \setminus A_t, B_t)$.

$$\nu = \prod_{k \in K} \delta_{A_k} \prod_{k \in K} \partial_{A_k} \tau$$

(Leray coboundary w.r.t. A_k)

(partial boundary w.r.t. A_k)

Thm (Picard, Lefschetz; Pham et al; MB, Panzer)

Let γ be a small loop around $L = \{0\}$, based at $t \in T$.

Then

$$\mathrm{Var}_\gamma : H_d(X \setminus A_t, B_t) \longrightarrow H_d(X \setminus A_t, B_t)$$

is given by

$$h \longmapsto \begin{cases} 0 & \text{if } d \neq n \\ N(h) \cdot \nu & \text{if } d = n \end{cases}$$

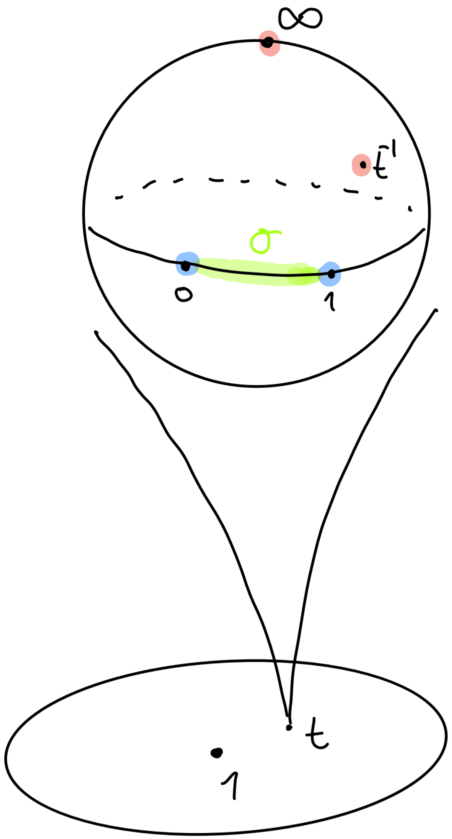
where

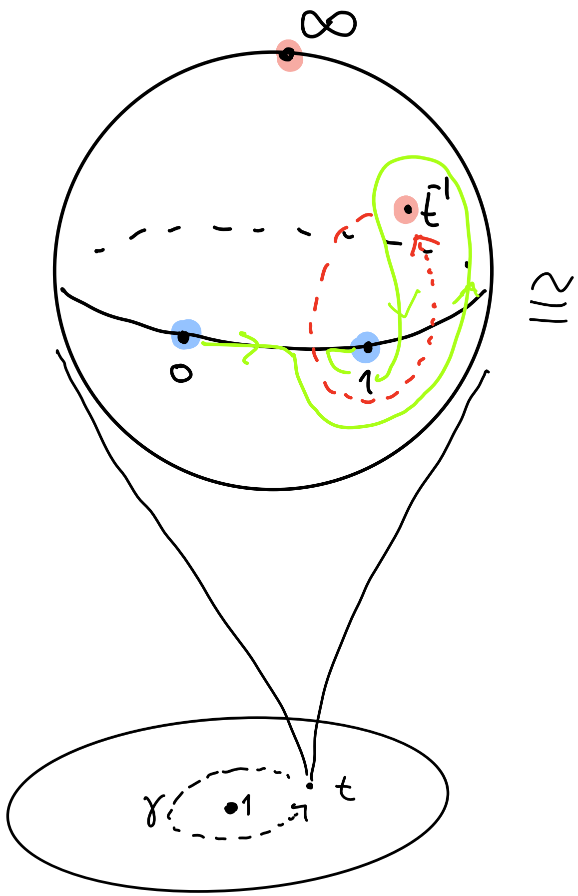
$$N(h) = \pm \left\langle \prod_{\ell \in L} \partial_{B_\ell} \zeta \;\middle|\; \prod_{\ell \in L} \partial_{B_\ell} h \right\rangle \in \mathbb{Z}$$

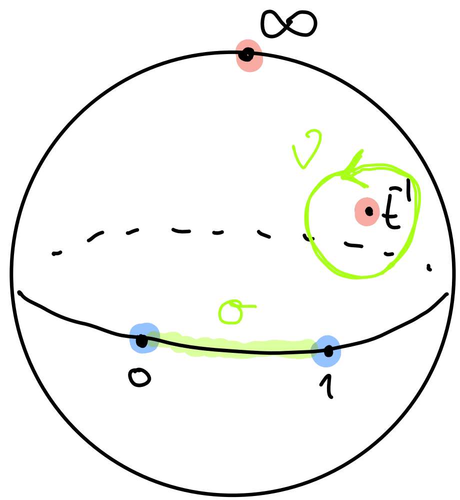
"relative intersection index"

Example

1) $I(t) = \log(1-t)$







$$\mathrm{Var}_{\circlearrowleft \cdot 1}\,\sigma = \pm \nu$$

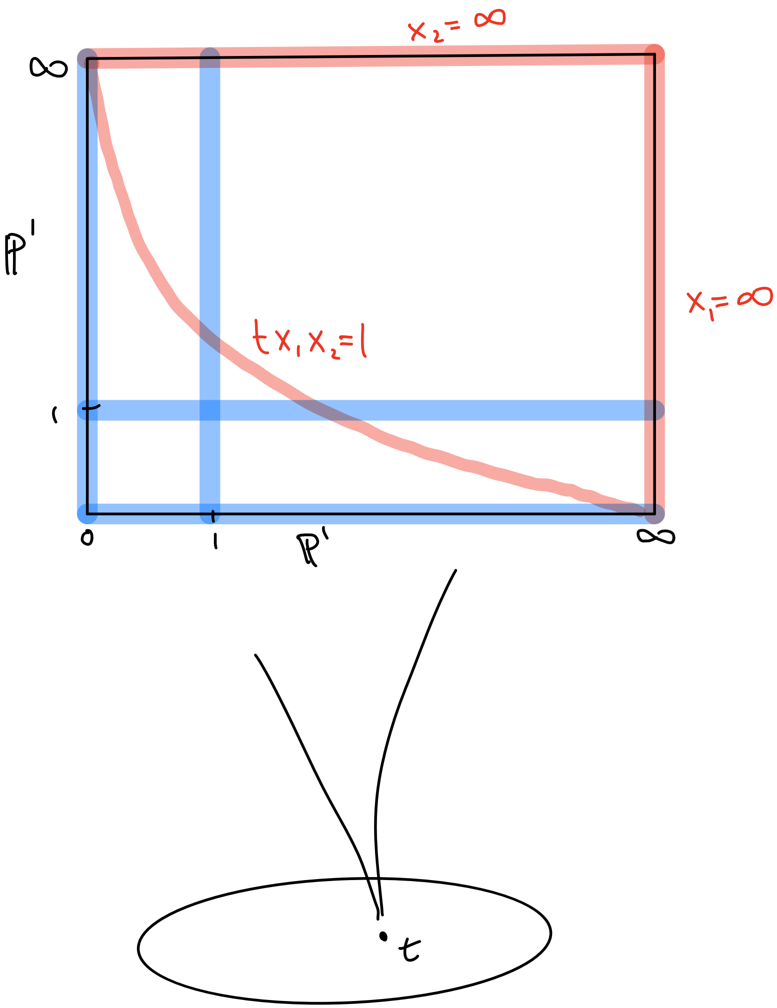
$$\log(1-t) = \int_\sigma \omega_t \quad \longmapsto \quad \int_{\sigma+\nu} \omega_t = \log(1-t) + 2\pi i$$

2) $$\mathrm{Li}_2(t) = \int_{[0,1]^2} \frac{t \, dx_1 \, dx_2}{1 - t x_1 x_2} ,$$

$X = \mathbb{P}^1 \times \mathbb{P}^1$, $T = \mathbb{P}^1$

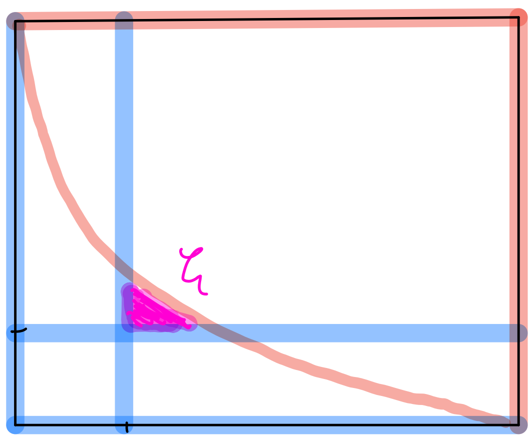
$A = \{ t x_1 x_2 = 1 \} \cup \{ x_1 = \infty \} \cup \{ x_2 = \infty \}$

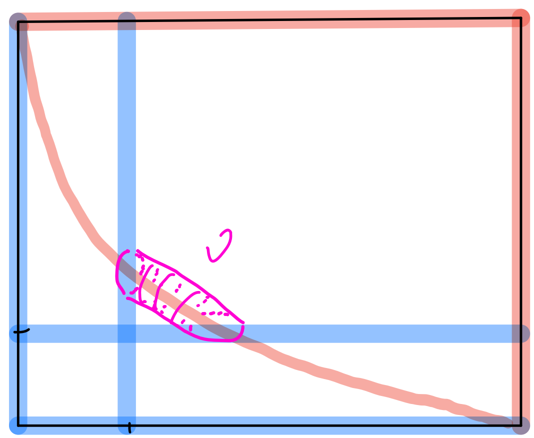
$B = \{ x_1 (1 - x_1) x_2 (1 - x_2) = 0 \}$



$\Rightarrow \quad L = \{ 0, 1, \infty \}$

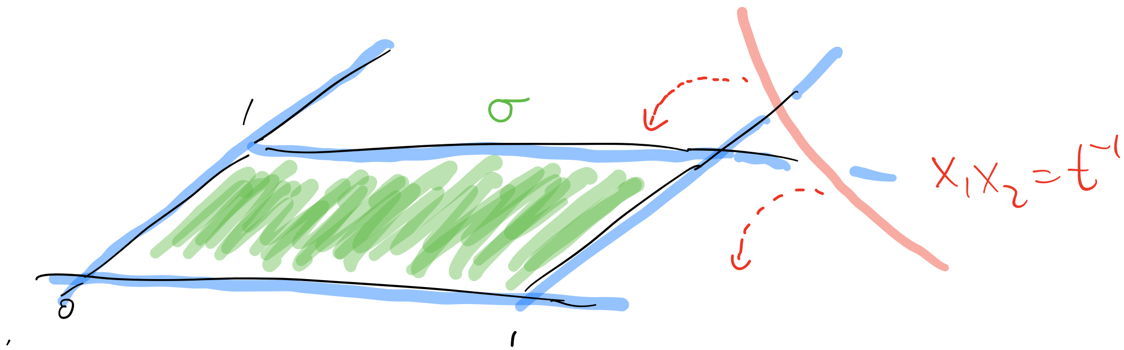
Vanishing cell and cycle for t close to 1:



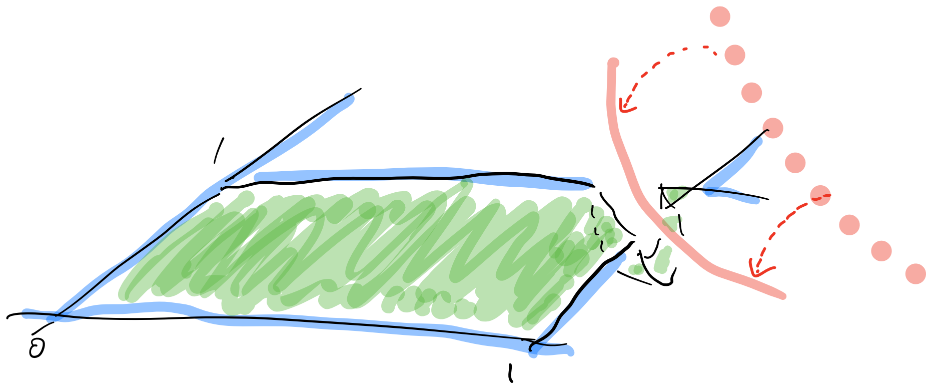


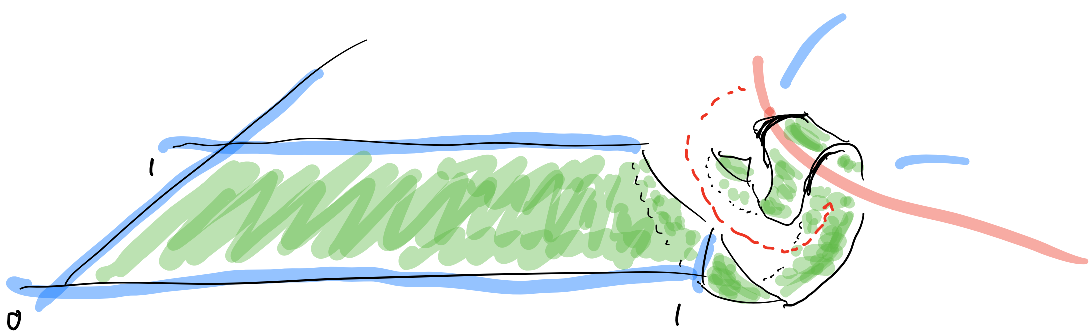
Now let t encircle 1:

$$\left(\mathrm{Li}_2(t) \longmapsto \mathrm{Li}_2(t) + 2\pi i \cdot \log(t) \right)$$

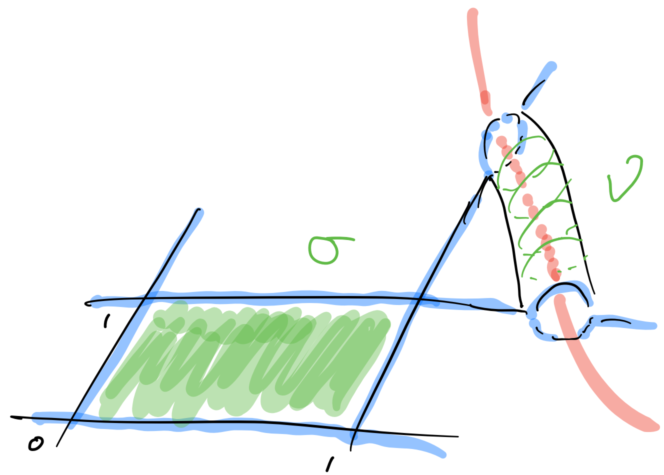


$$\mathrm{Var}_{\circlearrowleft}\,\sigma = \pm\nu$$





$\simeq$



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PL-Theorem gives a "complete" answer to Q2, but can we say something without computing all ingredients (which is difficult)?

Let $p \in \pi^{-1}(L)$ be critical for some $\pi|_{S_{KL}}$,

$$S_{KL} = \left(\bigcap_{k \in K} A_k\right) \cap \left(\bigcap_{\ell \in L} B_\ell\right).$$

<u>Def</u>

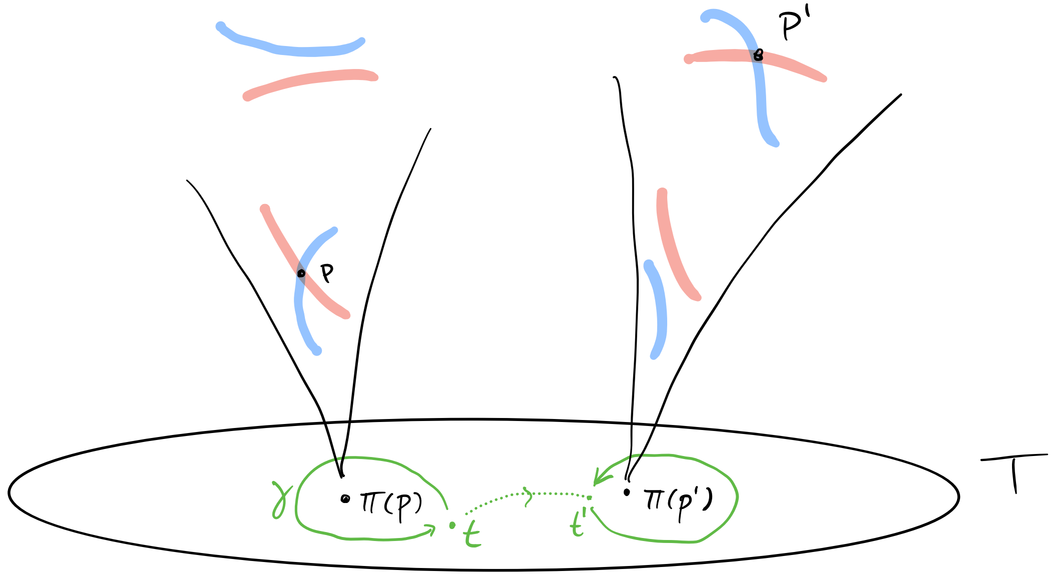
- p as above has <u>type</u> (K, L),
- For another critical point p' of type (K', L') we set $$p' \leq p \iff K' \supseteq K \wedge L' \subseteq L.$$

This defines a preorder on the set of ("simple pinch") critical points.

<u>Thm</u> (MB, Panzer) Let γ, γ' be small loops around $\pi(p)$ and $\pi(p')$, p of type (K,L), p' of type (K',L').

Then:

$$\mathrm{Var}_{\gamma'} \circ \mathrm{Var}_{\gamma} = 0 \quad \text{whenever} \quad p' \not\preceq p.$$



Proof

1) The vanishing cycle (w.r.t. p) is $\nu_p \in C_n\left(X \setminus \left(\bigcup_{k\in K} A_k\right)_t, \left(\bigcup_{\ell\in L} B_\ell\right)_t\right)$

PL-Thm: $\mathrm{Var}_{\gamma'}$ is determined by

$$N(\nu_p) = \pm \left\langle \prod_{\ell\in L'} \partial_{B_\ell} \zeta_{p'} \;\middle|\; \prod_{\ell\in L'} \underbrace{\partial_{B_\ell} \nu_p}_{=0 \text{ if } L' \notin L} \right\rangle$$

2) ν_p dies under the map (parallel transport)

$$H_n\left(X \setminus \left(\bigcup_{k\in K} A_k\right)_t, \left(\bigcup_{\ell\in L} B_\ell\right)_t\right) \to H_n\left(X \setminus \left(\bigcup_{k\in K'} A_k\right)_{t'}, \left(\bigcup_{\ell\in L'} B_\ell\right)_{t'}\right)$$

if $K \notin K'$.

Example

1) $\log(1-t)$

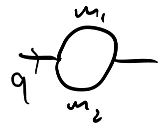
critical point	type
$p_0 = (\infty, 0)$	$(\{A_{xt=1}, A_\infty\}, \phi)$
$p_1 = (1,1)$	$(A_{xt=1}, B_1)$
$p_\infty = (0, \infty)$	$(A_{xt=1}, B_0)$

preorder:

$$
\begin{tikzcd}
p_\infty \arrow[dr, dash] & & p_1 \arrow[dl, dash] \\
& p_0 &
\end{tikzcd}
$$

$$
\begin{array}{l} p_\infty \nleq p_1 \\ p_1 \nleq p_\infty \end{array} \implies \mathrm{Var}_{\circlearrowleft i} \circ \mathrm{Var}_{\circlearrowleft j} = 0
$$

$i \neq j \in \{1, \infty\}$

2) 

critical point	type
$p_{m_2^2}$	$(A_{\mathcal{F}}, B_1)$
$p_{m_1^2}$	$(A_{\mathcal{F}}, B_2)$
p_+, p_-	$(A_{\mathcal{F}}, \phi)$
p_{q^2}	$(\{A_{\mathcal{F}}, A_{\mathcal{U}}\}, \phi)$

preorder:

$$
\begin{tikzcd}
p_{m_2^2} \arrow[dr, dash] & p_{m_1^2} \arrow[dl, dash] \\
p_+ \arrow[d, dash] & p_- \arrow[dl, dash] \\
p_{q^2} &
\end{tikzcd}
$$