Borel summability of the $1/N$ expansion in quartic O(N)-vector models

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Contents

Motivation: large N expansions in QFT

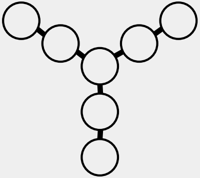
We are interested in a (Euclidean) quantum field theory with a global O(N) symmetry:

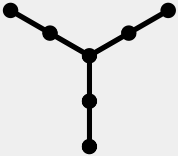
$$\begin{aligned}
\ln Z((g_k)_k, N) \text{ ``}=\text{''} & \ln \int_{\mathcal{S}'(\mathbb{R}^d)} [\mathcal{D}\phi] e^{-\int \left(\sum_i \phi_i(-\Delta+m^2)\phi_i + f(\|\phi\|^2, N) \right) d^d x} \\
\text{``}=\text{''} & \ln \int_{\mathcal{S}'(\mathbb{R}^d)} [\mathcal{D}\phi] e^{-\int \left(\sum_i \phi_i(-\Delta+m^2)\phi_i + \sum_{k\geq 2} \frac{g_k}{N^{k-1}} \|\phi\|^{2k} \right) d^d x} \\
\text{``}=\text{''} & \sum_G \prod_{k\geq 2} g_k^{n_k(G)} N^{1-\ell(G)} A(G) \\
\text{``}=\text{''} & \sum_{\ell\geq 0} N^{1-\ell} \ln Z_\ell((g_k)_k) \ ,
\end{aligned}$$

with $\ell(G) = 1 - F_{int}(G) + \sum_{k\geq 2}(k-1)V_k(G)$.

Intermediate field maps

Feynman graphs are in correspondence with Intermediate field maps





$$G \longleftrightarrow \mathcal{G}$$
$$\text{Vertex} \longleftrightarrow \text{Loop Edge}$$
$$\text{Edge} \longleftrightarrow \text{Corner}$$
$$F \longleftrightarrow \text{Loop Vertex}$$

$$\Rightarrow \ell(\mathcal{G}) = 1 - LV(\mathcal{G}) + LE(\mathcal{G}) = LE(\mathcal{G}) - (LV(\mathcal{G}) - 1) = \text{loop edge excess } (\mathcal{G}).$$

Intermediate field maps

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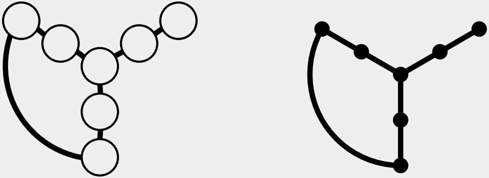

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The large N expansion for Matrices ['t Hooft 1974]

- Matrix integral:

$$Z((g_k)_k, 1/N) = \int [dM] e^{-N \mathrm{Tr}(\frac{M^2}{2} + \sum_{k\geq 2} g_k M^{2k})}$$
$$\text{``}=\text{''} \sum_G \prod_{k\geq 2} g_k^{n_k(G)} N^{V(G)-E(G)+F_{int}(G)} A(G)$$
$$\text{``}=\text{''} \sum_{g\geq 0} N^{2-2g} Z_g((g_k)_k)\,.$$

- In relation with (discretized) 2 dimensional quantum gravity:

$$Z(\Lambda, G) \text{``}=\text{''} \sum_{topologies} \int [\mathcal{D}g] e^{-\frac{1}{G}\int \sqrt{g}(R-2\Lambda)} \underset{D=2}{=} \sum_{g\geq 0} e^{-\frac{1}{G}(2-2g-2\Lambda Vol(\mathcal{M}))}.$$

Beyond 2 dimensions: the large N expansion for Tensors [Gurau '11]

- For the O(N)-vector models, only one connected invariant: $\|\phi\|^2$, and for the $O(N)^{\otimes 2}$-matrix models, one per integer k: $\mathrm{Tr} M^k$.
- For $D \geq 3$, there are much more $O(N)^{\otimes D}$ invariants $\mathcal{B}$. They are indexed by D-coloured graphs.
- Tensor integral:

$$Z((g_{\mathcal{B}})_{\mathcal{B}}, 1/N) = \int [dT] e^{-N^{D-1} \sum_{\mathcal{B}, \omega(\mathcal{B})=0} g_{\mathcal{B}} \mathrm{Tr}_{\mathcal{B}}(T)}$$

$$\text{“} = \text{”} \sum_{G} \prod_{\mathcal{B}} g_{\mathcal{B}}^{n_{\mathcal{B}}(G)} N^{(D-1)(V(G)-E(G))+F_{int}(G)} A(G)$$

$$\text{“} = \text{”} \sum_{\omega \geq 0} N^{D-\omega} \ln Z_{\omega}((g_{\mathcal{B}})_{\mathcal{B}}) \,.$$

- The dominant graphs are the graphs such that $\omega = 0$: they maximize the number of faces. They are called *melonic graphs*.

The quartic O(N) vector model

- We define it as a perturbed Gaussian measure (there is no Lebesgue measure on $\mathcal{S}'(\mathbb{R}^d)$).
- Let $N \in \mathbb{N}_{\geq 1}$ and $g \in \{z \in \mathbb{C} \mid \Re z > 0\}$:

$$\mathbb{E}[F(\phi)] = \frac{\int d\mu_{I_N}(\phi) e^{-\frac{g}{8N}\|\phi\|^4} F(\phi)}{\int d\mu_{I_N}(\phi) e^{-\frac{g}{8N}\|\phi\|^4}} .$$

- Laplace transform of the measure (partition function with sources $J \in \mathbb{R}^N$):

$$Z(g, \frac{1}{N}; J) = \int d\mu_{I_N}(\phi) e^{-\frac{g}{8N}\|\phi\|^4} \times \mathbb{E}[e^{\sqrt{N}\langle J,\phi\rangle}] = \int d\mu_{I_N}(\phi) e^{-\frac{g}{8N}\|\phi\|^4 + \sqrt{N}\langle J,\phi\rangle} .$$

- Free energy (log of the normalisation constant):

$$W(g, 1/N) := \ln Z(g, 1/N; 0) = \ln \int d\mu_{I_N}(\phi) e^{-\frac{g}{8N}\|\phi\|^4}$$

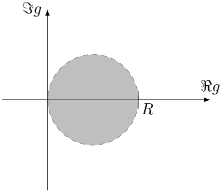
.

Borel summable series and Borel summable functions

- Let $A(z) = \sum_{k\geq 0} a_k z^k$ a formal power series.
- If $B(t) = \sum_{k\geq 0} \frac{a_k}{k!} t^k$ is absolutely convergent in the disk $\mathcal{D}(0, \sigma^{-1})$ and can be analytically continued to the strip $\{t : |\Im t| < \sigma^{-1}\}$ where it obeys the following bound: $|B(t)| \leq e^{\frac{\Re t}{R}}$ with $R > 0$.
- Then the series $B(t)$ is the *Borel transform* of A and the *Borel sum* of A, $f(z) := \frac{1}{z}\mathcal{L}(B)(\frac{1}{z})$, is analytic in the disk $\{z : \Re\frac{1}{z} < \frac{1}{R}\}$.
- A function $f(z)$ analytic in the disk $\{z : \Re\frac{1}{z} < \frac{1}{R}\}$ and endowed with a (perhaps divergent) asymptotic series in 0 such that its Taylor rest term in 0 is of at most factorial blow up is a *Borel summable function*.
- They are in one-to-one correspondance: the Borel sums of Borel summable series are Borel summable functions and the asymptotic series of Borel summable functions are Borel summable series [Sokal 1979].

Uniform (in w) Borel summability of a function

The Nevalinna-Sokal theorem (1979) for $f : \mathbb{C}^2 \to \mathbb{C}$, $(g, w) \mapsto f(g, w)$



+ *uniform in w* bound on the rest:

$$\left| f(g, w) - \sum_{k=0}^{q-1} a_k(w) g^k \right| \leq C K^q |z|^q q! \,,$$

$\Rightarrow$ f is Borel summable in g uniformly in w.

$$f(z, w) = \frac{1}{z} \int_{\mathbb{R}_+} e^{-t/z} \left(\sum_{k=0}^{\infty} \frac{a_k(w)}{k!} t^k \right) dt \,.$$

The Loop Vertex Expansion (LVE) [Rivasseau '07]

- A new expansion of the connected correlation functions (logarithmic in the partition function).
- Relies on the intermediate field transformation:

$$e^{-\frac{x^2}{2}} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dy \; e^{-\frac{y^2}{2} + \imath xy} = \int d\mu_1(y) \; e^{\imath xy} \; ,$$

- the replica trick:

$$\int e^{-\frac{x^2}{2}} f^n(x) dx = \int e^{-\frac{\sum_{ij} X_i X_j}{2}} f^{\otimes n}(X) d^n X = \int e^{-\frac{\sum_{ij} X_i X_j}{2}} \prod_i f(X_i) d^n X \; ,$$

- and the BKAR formula.

The Loop Vertex Expansion (LVE) [Rivasseau '07]

The BKAR formula [Abdesselam & Rivasseau 1995]

Let $f : (x_\ell)_{\ell \in K_n} \mapsto f(x_\ell)$ be a smooth function. We have

$$f(\mathbb{1}) = \sum_{F \in \mathcal{F}_n} \left[\prod_{T \in F} \int du_T \partial_T \right] f\Big(\sum_T W^T(u) \otimes 0_{\hat{T}} \Big) ,$$

where $W^T_{ij}(u) = W^T_{ji}(u) = \inf_{(k,l)\in P^T_{i\leftrightarrow j}} u_{kl}$ (in particular $W^T_{ii}(u) = 1$),
$\int du_T = \prod_{(k,l)\in T} du_{kl}$ and $\partial_T = \prod_{(k,l)\in T} \partial_{x_{kl}}$.
Therefore, if $f(X) = e^{\frac{1}{2}\langle X | M \rangle}$ (for us f will typically be $e^{\frac{1}{2}\langle X | \partial_\phi C \partial_\phi \rangle}$),

$$f(\mathbb{1}) = \sum_{F \in \mathcal{F}_n} \prod_{T \in F} \int du_T \prod_{(k,l)\in T} M_{kl}\, e^{\frac{1}{2}\langle W^T(u) | M^T \rangle} = \sum_{F \in \mathcal{F}_n} \prod_{T \in F} A_T .$$

The Loop Vertex Expansion (LVE)

- The LVE was adapted to the study of a quantum field theory with multiscale analysis (MLVE).
- Results in the expansion of the cumulants as a sum over trees that is convergent thanks to the slower proliferation of trees in $O(1)^n n!$ compared to Feynman graphs in $O(1)^n n!^2$.
- The outcome is a domain of analyticity and a bound on the rest term in 0.

Main results

Theorem 1: analyticity domain of the free energy and the cumulants.

The previous series defines an analytic function of the two complex variables g and ε in the following subdomain of $\mathbb{C}^2$:

$$\mathfrak{C} = \left\{ \begin{pmatrix} g = |g| e^{\imath\varphi} \\ \varepsilon = |\varepsilon| e^{\imath\theta} \end{pmatrix} \in \mathbb{C}^2 \;\middle|\; \exists \psi, \begin{cases} |g| < \frac{1}{4}(1 + \cos(\varphi + \psi))\sqrt{\cos(\psi - \theta)} \\ |\varphi + \psi| < \pi \\ |\psi - \theta| < \frac{\pi}{2} \end{cases} \right\}$$

At fixed g, the induced analyticity domain in the ε-plane is a Riemann sheet, independent of the modulus of ε, and where its argument can evolve in a range $-\frac{3\pi}{2} - \varphi < \theta < \frac{3\pi}{2} - \varphi$. In particular, as soon as g is non negative, it always includes a (Sokal) disk of any radius tangent to the imaginary axis in 0.

Main results

Theoreom 2: Borel summability in $1/N$ of the FE and the cumulants.

This series is Borel summable in ε along the positive real axis uniformly in g for all g in a slightly restricted subdomain of $\mathfrak{C}$:

$$\mathfrak{C}_\alpha = \left\{ (g,\varepsilon) \in \mathbb{C}^2 \;\middle|\; \exists \psi, \begin{cases} |g| \leq \frac{1}{4}(1+\cos(\varphi+\psi))\sqrt{\cos(\psi-\theta)}\,(1-\alpha) \\ |\varphi+\psi| \leq \pi - \alpha \\ |\psi-\theta| \leq \frac{\pi}{2} - \alpha \end{cases} \right\}$$

The cumulants can therefore be computed as the Borel sum of their large N expansion.

LVE of the quartic O(N)-vector model

Let $g = |g|e^{\imath\varphi}$, $\varepsilon = |\varepsilon|e^{\imath\theta}$ and $\psi \in (-\pi, \pi)$:

- The intermediate field (interpolation in $1/N$ [Kupiainen 1980]):

$$Z(g,\varepsilon) = \int d\mu_{I_N}(\phi) e^{-\frac{\varepsilon g}{8}\|\phi\|^4} = \int d\mu_{I_N}(\phi) d\mu_1(\sigma) e^{\imath \frac{\sqrt{\varepsilon g}}{2}\sigma \|\phi\|^2}$$
$$= \int d\mu_1(\sigma) e^{\frac{1}{2\varepsilon} \ln R(\sqrt{\varepsilon}\sigma, g)} = \int d\mu_\varepsilon(\sigma) e^{\frac{1}{2\varepsilon} \ln R(\sigma, g)} \quad \left(R(\sigma, x) := \frac{1}{1 - \imath\sqrt{x}\sigma}\right)$$

- To obtain a maximal domain of analyticity define:

$$Z_\psi(g,\varepsilon) = \int_{e^{\imath\frac{\psi}{2}}\mathbb{R}} d\mu_\varepsilon(\sigma) e^{\frac{1}{2\varepsilon}\ln R(\sigma,g)} = \int d\mu_{e^{-\imath\psi}\varepsilon}(\sigma) e^{\frac{1}{2\varepsilon}\ln R(\sigma, e^{\imath\psi} g)}$$
$$= \int d\mu_{e^{-\imath\psi}\varepsilon}(\sigma) \sum_{n\geq 0} \frac{1}{(2\varepsilon)^n n!} (\ln R(\sigma, e^{\imath\psi} g))^n .$$

- Legal change of order of summation and integration:

$$\sum_{n\geq 0} \frac{1}{(2\varepsilon)^n n!} \int d\nu_{e^{-\imath\psi}\varepsilon}(\sigma) (\ln R(\sigma, e^{\imath\psi} g))^n .$$

LVE of the quartic O(N)-vector model

- The copies trick:

$$Z(g,\varepsilon) = \sum_{n\geq 0} \frac{1}{(2\varepsilon)^n n!} \int d\nu_{e^{-\imath\psi}\varepsilon}(\sigma) (\ln R(\sigma, e^{\imath\psi} g))^n$$

$$= \sum_{n\geq 0} \frac{1}{(2\varepsilon)^n n!} \int d\mu_{e^{-\imath\psi}\varepsilon\mathbb{1}}(\sigma) \prod_{i=1}^{n} \ln R(\sigma^{(i)}, e^{\imath\psi} g)\,.$$

- The BKAR formula:

$$Z(g,\varepsilon) = \sum_{n\geq 0} \frac{1}{(2\varepsilon)^n n!} \sum_{F\in\mathcal{F}_n} (\imath\sqrt{g})^{\sum_i d_i} \varepsilon^{|E(F_n)|}$$
$$\times \int du_F \int d\mu_{e^{-\imath\psi}\varepsilon W^F(u)}(\sigma) \prod_{i=1}^{n} (d_i-1)! R^{d_i}(\sigma^{(i)}, e^{\imath\psi} g)\,.$$

so that the partition functions factors over the trees $\Rightarrow$ its logarithm rewrites as a sum over the trees.

LVE of the quartic O(N)-vector model

- Mixed expansion of the free energy:

$$\begin{aligned}\varepsilon \ln Z(g,\varepsilon) &= \sum_{n\geq 1} \frac{(-g/2)^{n-1}}{2n!} \sum_{T\in\mathcal{T}_n} \int du_T \\ &\quad \times \left[e^{\frac{\varepsilon}{2e^{\imath\psi}}\langle\partial,\partial\rangle_{W^T(u)}} \prod_{i=1}^{n} (d_i-1)! R^{d_i}(\sigma^{(i)}, e^{\imath\psi} g) \right]_{\sigma=0} \\ &= \sum_{n\geq 1} \frac{(-g/2)^{n-1}}{2n!} \sum_{T\in\mathcal{T}_n} \int du_T \sum_{\ell\geq 0} \frac{1}{\ell!} \left(\frac{\varepsilon}{2e^{\imath\psi}} \langle\partial,\partial\rangle_{W^T(u)} \right)^{\ell} \\ &\quad \times \prod_{i=1}^{n} (d_i-1)! R^{d_i}(\sigma^{(i)}, e^{\imath\psi} g) \\ &\text{“}=\text{”} \sum_{\ell\geq 0} \varepsilon^{\ell} \ln Z_{\ell}(g) \,.\end{aligned}$$

- This is the $1/N$ expansion of the logarithm of the logarithm of the partition function!

Combinatorial core of the proof

Integration over $\mathbb{R}^N \Leftrightarrow$ "0 dimensional QFT" $\Rightarrow$ to disentangle analysis from combinatorics $\Rightarrow$ analysis is trivial (the resolvent $R(\sigma, z)$ is simply bounded by $1/|\cos(\arg z/2)|$, the Gaussian integration by $1/\cos^{n/2}(\arg \varepsilon - \psi)$).

Combinatorics of the BS of the $1/N$ expansion: the rest term of $\ln Z$ is of order

$$
\begin{aligned}
R_\ell(\ln Z) &\sim \frac{1}{\ell!}\sum_{n\geq 1}\frac{(|g|/2)^n}{n!}\sum_{T\in\mathcal{T}_n}\underbrace{(2n-2)(2n-2+1)\dots(2n-2+2\ell-1)}_{\text{choices of the corners where to add the } 2\ell \text{ half-edges}}\prod_{i=1}^n (d_i-1)! \\
&\sim \frac{1}{\ell!}\sum_{n\geq 1}\frac{(|g|/2)^n}{n!}\sum_{\substack{d_1,\dots,d_n\\ \sum_i d_i = 2n-2}}\frac{(n-2)!}{\prod_{i=1}^n (d_i-1)!}\,\frac{(2n-2+2\ell-1)!}{(2n-3)!}\prod_{i=1}^n (d_i-1)! \\
&\sim \frac{(2\ell)!}{\ell!}\sum_{n\geq 1}(|g|/2)^n\frac{(n-2)!}{n!}\binom{2n-3}{n-1}\binom{2n-3+2\ell}{2n-3} \\
&\sim \ell!\sum_{n\geq 1}(K|g|)^n\,.
\end{aligned}
$$

Summary and outlooks

- The free energy and the cumulants of the quartic O(N)-vector model are analytic in both the coupling constant and $1/N$ in a domain of the complex plane.
- If the coupling constant is non-negative and of small enough modulus, their $1/N$ expansion is Borel summable along the real axis.
- The next step is to study a QFT, with some renormalisation.
- It should be doable in 2 dimensions, but the LVE was never studied in 3 dimensions.

$$Z(g,\varepsilon) = e^{\frac{g}{4}(1+2\varepsilon)C_\rho(0)^2\Lambda} \int d\mu_{\varepsilon I}(\sigma) e^{-\frac{1}{2\varepsilon}\mathrm{Tr}\ln_2(1-i\sqrt{g}C_\rho\sigma)-\imath\sqrt{g}\mathrm{Tr}(C_\rho\sigma)}$$

- What about the large N expansion for Matrices/Tensors?

Thank you for the attention!