Generalized symmetries as homotopy Lie algebras

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What do we know about physics at quantum gravity scales?

- Appropriate space-time probe is not a point particle.
- Concept of symmetry needs to be generalized.
- Field content needs to be extended.
- Wilsonian separation of scales (probably) fails in QG regime.

Different aspects made precise in string theory, holography, matrix/tensor models....

What are the relevant symmetries at quantum gravity scales?

In string theory space-time probe is spatially extended, eg strings and branes

- $\rightsquigarrow$ can wrap compactified spaces $\rightsquigarrow$ dualities in string theory, eg T-duality
- $\rightsquigarrow$ couple to higher gauge potential $\rightsquigarrow$ higher gauge theories
- $\rightsquigarrow$ low-energy effective dynamics $\rightsquigarrow$ non-commutative/non-associative deformation

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$\rightsquigarrow$ low-energy effective dynamics $\rightsquigarrow$ non-commutative/non-associative deformation

$\rightsquigarrow$ natural framework for the description of generalized symmetries in the quantum gravity regime $\rightsquigarrow$ homotopy Lie algebras (L_∞-algebras), Stasheff '63, Stasheff, Schlesinger '77

In this talk

GOAL

Demonstrate that L_∞-algebra is natural framework for the description of generalized symmetries.

PLAN

- L_∞-algebra - short introduction
- L_∞-algebra - coalgebra formulation
- Drinfel'd twist of L_∞-algebra
- DFT algebroid

L_∞-algebra

On L_∞-algebras

L_∞-algebra $\rightsquigarrow$ generalization of differential graded Lie algebras encoding both the kinematics and dynamics of a given field theory.

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- Quantization
  - BV formalism $\sim$ L_∞-algebra Zwiebach '92
  - Deformation quantization: formality thm $\sim$ L_∞ quasi-isomorphism Kontsevich '97;
    Poisson sigma model quantization Cattaneo, Felder, '99

On L_∞-algebras

L_∞-algebra $\rightsquigarrow$generalization of differential graded Lie algebras encoding both the kinematics and dynamics of a given field theory.

- Quantization $\rightsquigarrow$ BV-BRST, deformation quantization
- Geometry
  - Graded geometry: L_∞-algebra (cyclic) $\equiv$ Q(P) manifolds AKSZ '95
  - Generalized geometry of Courant, double field theory and exceptional algebroids
    Roytenberg, Weinstein '98; Deser, Saemann '16, LJ, Grewcoe '20; Cederwall, Palmkvist '18

On L_∞-algebras

L_∞-algebra $\rightsquigarrow$ generalizations of differential graded Lie algebras.

- Quantization $\rightsquigarrow$ BV-BRST, deformation quantization
- Graded and generalized geometry
- NC/NA field theory and gravity
  - $\star$-product: bootstraping nc gauge theories using L_∞ Blumenhagen et al '18
  - Drinfel'd twist and braided L_∞ Dimitrijević Ćirić et al '21, Nguyen, Schenkel, Szabo '21
  - HS in unfolded formalism Vasiliev

Algebra formulation

Def1: An L_∞-algebra (L, μ_i) is a $\mathbb{Z}$-graded vector space $\mathsf{L} = \bigoplus_k \mathsf{L}_k$ with a collection of graded totally antisymmetric multilinear maps

$$\mu_i : \underbrace{\mathsf{L} \times \cdots \times \mathsf{L}}_{i\text{-times}} \to \mathsf{L}.$$

of degree $2 - i$ which satisfy the homotopy Jacobi identities:

$$\sum_{j+k=n} \sum_{\sigma} \chi(\sigma; l_1, \ldots, l_n)(-1)^k \mu_{k+1}(\mu_j(l_{\sigma(1)}, \ldots, l_{\sigma(j)}), l_{\sigma(j+1)}, \ldots, l_{\sigma(n)}) = 0;$$

$$\forall n \in \mathbb{N}_0 \quad \forall l_i \in \mathsf{L}.$$

$\chi(\sigma; l_1, \ldots, l_n)$ is the graded Koszul sign that includes the sign from the parity of the permutation σ ordered as: $\sigma(1) < \cdots < \sigma(j)$ and $\sigma(j+1) < \cdots < \sigma(n)$.

Algebra formulation continued

$n = 0$: $\mu_1 \mu_0 = 0$

$n = 1$: $\mu_1^2(l) = \mu_2(\mu_0, l)$

$n = 2$: $\mu_1(\mu_2(l_1, l_2)) - \mu_2(\mu_1(l_1), l_2) - (-1)^{1+|l_1||l_2|}\mu_2(\mu_1(l_2), l_1) = -\mu_3(\mu_0, l_1, l_2)$

$$\begin{aligned} n = 3: \ & \mu_1(\mu_3(l_1, l_2, l_3)) - \mu_2(\mu_2(l_1, l_2), l_3) - (-1)^{|l_2||l_3|}\mu_2(\mu_2(l_1, l_3), l_2) - \\ & - (-1)^{|l_1|(|l_2|+|l_3|)}\mu_2(\mu_2(l_2, l_3), l_1) + \mu_3(\mu_1(l_1), l_2, l_3) - \\ & - (-1)^{|l_1||l_2|}\mu_3(\mu_1(l_2), l_1, l_3) + (-1)^{|l_3|(|l_1|+|l_2|)}\mu_3(\mu_1(l_3), l_1, l_2) = \mu_4(\mu_0, l_1, l_2, l_3), \end{aligned}$$

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\end{aligned}
$$

Given $(\mu_0), \mu_1, \mu_2$ the rest can be “bootstrapped”.

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$n = 3$: $\rightsquigarrow$ generalized Jacobi identity for bracket μ_2 controlled by higher brackets

When $\mu_0 = 0$, μ_1 is differential on L and a derivation of (super)bracket μ_2. The elements of graded vector space $\mathsf{L} = \bigoplus_k \mathsf{L}_k$ form a cochain complex

$$\cdots \xrightarrow{\mu_1} \mathsf{L}_k \xrightarrow{\mu_1} \mathsf{L}_{k+1} \xrightarrow{\mu_1} \cdots$$

Algebra formulation continued

Ex1: $\mu_i = 0$ for $i \neq 2 \rightsquigarrow$ graded Lie algebra with μ_2 being graded Lie bracket.

Ex2: $\mu_i = 0$ for $i \geq 3 \rightsquigarrow$ curved differential graded Lie algebra.

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Cdgla: Graded Lie algebra $\mathfrak{g}$, derivation d with degree 1, and curvature R of degree 2 s.t.

$$dR = 0 \ , \quad d^2 x = [R, x], \forall x \in \mathfrak{g} \ .$$

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L_∞: Graded vector space L with maps $R \rightsquigarrow \mu_0$, $d \rightsquigarrow \mu_1$, the graded Lie bracket $\rightsquigarrow \mu_2$, satisfying the homotopy relations

$$\mu_1 \mu_0 = 0 \ , \quad \mu_1(\mu_1(l)) = \mu_2(\mu_0, l), l \in \mathrm{L} \ .$$

Leibniz: $\mu_1(\mu_2(l_1, l_2)) = \mu_2(\mu_1(l_1), l_2) + (-1)^{1+|l_1||l_2|}\mu_2(\mu_1(l_2), l_1)$

Jacobi: $\mu_2(\mu_2(l_1, l_2), l_3) + (-1)^{|l_2||l_3|}\mu_2(\mu_2(l_1, l_3), l_2) + (-1)^{|l_1|(|l_2|+|l_3|)}\mu_2(\mu_2(l_2, l_3), l_1) = 0$

Algebra formulation continued

Maurer-Cartan (MC) elements of L_∞-algebra are elements of homogeneous subspace L_1 satisfying

$$\mu_0 + \mu_1(x) + \frac{1}{2!}\mu_2(x,x) + \frac{1}{3!}\mu_3(x,x,x) + \cdots = 0 \ , x \in \mathsf{L}_1$$

generalised MC equation.

Algebra formulation continued

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generalised MC equation.

L_∞-morphism between two L_∞-algebras (L,μ) and (L',μ') is a collection of multilinear, totally graded antisymmetric homogeneous maps $\phi_i : \mathrm{L}^{\times i} \to \mathrm{L}'$ of degree $1-i$, $i \in \mathbb{N}_0$ if...

$$\phi_1(\mu_1(l)) = \mu_1'(\phi_1(l))$$

$$\phi_1(\mu_2(l_1,l_2)) - \phi_2(\mu_1(l_1),l_2) + (-1)^{|l_1||l_2|}\phi_2(\mu_1(l_2),l_1) = \mu_1'(\phi_2(l_1,l_2)) + \mu_2'(\phi_1(l_1),\phi_1(l_2))$$

...

L_∞ - perturbative field theory - sketch

Hohm, Zwiebach '17; Jurčo et al. '18, '20; Giotopoulos, Szabo '21

- $L_\infty \rightsquigarrow$ symmetry (gauge) algebra
- MC equations $\rightsquigarrow$ eoms

$$\sum_i \frac{1}{i!}\mu_i(x,\ldots,x) = 0 \ , x \in L_1$$

$$\delta_c x = \sum_i \frac{1}{i!}\mu_{i+1}(x,\ldots,x,c) \ , c \in L_0$$

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To define a classical action and/or solution of classical master equation we need:

- Tensor product (of complexes)
- Inner product $\rightsquigarrow$ cyclic L_∞

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$$S_{\mathrm{MC}}[x] \equiv \sum_{i \geqslant 0} \frac{1}{(i+1)!}\langle x, \mu_i(x, \ldots, x)\rangle .$$

L_∞ - perturbative field theory - sketch

Hohm, Zwiebach ‘17; Jurčo et al. ‘18, ‘20; Giotopoulos, Szabo ’21

- quasi-isomorphisms $\rightsquigarrow$ equivalent field theories
- homotopy transfer $\rightsquigarrow$ effective field theories
- minimal model thm $\rightsquigarrow$ tree level amplitudes
- quantum homotopy $\rightsquigarrow$ loops c.f. Brano talk
- renormalization Costello; Chiaffrino, Sachs ’21, c.f. Kasia talk

L_∞ - coalgebra formulation

Lada, Stasheff '92, Lada, Markl '94

An L_∞ algebra is a $\mathbb{Z}$-graded vector space

$$X = \bigoplus_{d \in \mathbb{Z}} X_d$$

with multilinear graded symmetric maps $b_i : X^{\otimes i} \to X$ of degree 1 such that the coderivation $D = \sum_{i=0} b_i$ is nilpotent.

$$\begin{aligned} D^2 = 0 : & b_1(b_0) = 0 \ , \\ & b_2(b_0, x) + b_1^2(x) = 0 \ , \\ & b_3(b_0, x_1, x_2) + b_2(b_1(x_1), x_2) + (-1)^{|x_1||x_2|} b_2(b_1(x_2), x_1) + b_1(b_2(x_1, x_2)) = 0 \ . \end{aligned}$$

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Suspension or shift isomorphism s

$$s : L \to L[1] \ \text{ s.t. } \ (L[1])_d = X_{d+1} \ ,$$

induces isomorphism of graded algebras and décalage isomorphism of brackets

$$\mu_i = (-1)^{\frac{1}{2} i(i-1)+1} s^{-1} \circ b_i \circ s^{\otimes i} \ .$$

L_∞ - coalgebra formulation

Extend the maps b_i on the whole graded symmetric tensor algebra over field $K =: S^0 X$

$$\mathrm{S}(X) := \bigoplus_{n=0}^{\infty} S^n X \ ,$$

with graded symmetric tensor product $\vee$.

The maps $b_i : S^j X \to S^{j-i+1} X$ act as a coderivation:

$$b_i(x_1 \vee \ldots \vee x_j) = \sum_{\sigma \in \mathrm{Sh}(i,j-i)} \epsilon(\sigma; x) b_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}) \vee x_{\sigma(i+1)} \vee \ldots \vee x_{\sigma(j)} \ , j \geq i \ ,$$

where $\epsilon(\sigma; x)$ is the Koszul sign, and $\mathrm{Sh}(p, m-p) \in S_m$ denotes those permutations ordered as $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(m)$.

Introducing the permutation map $\tau^\sigma : X^{\otimes i} \to X^{\otimes i}$

$$b_i \circ \mathrm{id}^{\vee j} = \sum_{\sigma \in \mathrm{Sh}(i,j-i)} (b_i \vee \mathrm{id}^{\vee(j-i)}) \circ \tau^\sigma \ , \qquad j \geq i \ .$$

L_∞ - coalgebra formulation

Degree 1 coderivation $D : \mathrm{S}(X) \to \mathrm{S}(X)$ satisfies the co-Leibniz property:

$$\Delta \circ D = (1 \otimes D + D \otimes 1) \circ \Delta \ ,$$

with the coproduct map map $\Delta : \mathrm{S}(X) \to \mathrm{S}(X) \otimes \mathrm{S}(X)$

$$\Delta \circ \mathrm{id}^{\vee m} = \sum_{p=0}^{m} \sum_{\sigma \in \mathrm{Sh}(p,m-p)} (\mathrm{id}^{\vee p} \otimes \mathrm{id}^{\vee (m-p)}) \circ \tau^{\sigma} \ , \ p, m \geq 0 \ ,$$

L_∞ - coalgebra formulation

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$\rightsquigarrow$ L_∞-algebra as a coalgebra with coderivation and counit $\varepsilon : \mathsf{S}(X) \to K$, where $\varepsilon(1) = 1$ and $\varepsilon(x) = 0, \ x \in X$.

$L_\infty \rightsquigarrow$ Hopf algebra

Graded symmetric tensor algebra $\mathrm{S}(X)$ has an algebra structure given by the graded symmetric tensor product $\vee$ and a unit map $\eta : K \to \mathrm{S}(X)$, where $\eta(1) = 1$.

The algebra and coalgebra structure on $\mathrm{S}(X)$ make up a bialgebra, that admits a graded antipode map S

$$S(x_1 \vee \cdots \vee x_m) = (-1)^m (-1)^{\sum_{i=2}^m \sum_{j=1}^{i-1} |x_i||x_j|} x_m \vee \cdots \vee x_1 \ .$$

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$\rightsquigarrow$ the homotopy Lie algebra defined by the coalgebra structure on the graded symmetric tensor space $\mathsf{S}(X)$ extends to a cocommutative and coassociative Hopf algebra with compatible coderivation. Grewcoe, LJ, Kodzoman, Manolakos, '22

Non-commutative deformation

$\rightsquigarrow$ introduce non-(co)commutative deformation using Drinfel'd twist.

Using invertible twist element $\mathcal{F} =: f^{\alpha} \otimes f_{\alpha} \in H \otimes H$

$$(\mathcal{F} \otimes 1)(\Delta \otimes id)\mathcal{F} = (1 \otimes \mathcal{F})(id \otimes \Delta)\mathcal{F} \ ,$$
$$(\epsilon \otimes id)\mathcal{F} = 1 \otimes 1 = (id \otimes \epsilon)\mathcal{F} \ ,$$

we obtain $(H^{\mathcal{F}}, \vee, \Delta^{\mathcal{F}}, S^{\mathcal{F}}, \epsilon)$, where $H^{\mathcal{F}}$ is the same as H as vector spaces and:

$$\Delta^{\mathcal{F}}(h) = \mathcal{F}\Delta(h)\mathcal{F}^{-1}, \ h \in H \ ,$$

and $S^{\mathcal{F}} = S$ for Abelian twist.

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we obtain $(H^\mathcal{F}, \vee, \Delta^\mathcal{F}, S^\mathcal{F}, \epsilon)$, where $H^\mathcal{F}$ is the same as H as vector spaces and:

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$\rightsquigarrow$ twisted L_∞ or $(L_\infty^\mathcal{F}, \vee, \Delta^\mathcal{F}, S, \epsilon)$

Drinfel'd twist

In the spirit of deformation quantisation, while twisting Hopf algebra we simultaneously twist its modules. Taking Hopf algebra L_∞ as its own module
$\rightsquigarrow (L_\infty^\star, \vee_\star, \Delta_\star, S_\star, \epsilon)$:

$$
\begin{aligned}
x_1 \vee_\star x_2 &= \bar{f}^\alpha(x_1) \vee \bar{f}_\alpha(x_2) \ , \\
\Delta_\star(x) &= x \otimes 1 + \bar{R}^\alpha \otimes \bar{R}_\alpha(x) \ , \\
S_\star(x) &= -\bar{R}^\alpha(x)\bar{R}_\alpha \ .
\end{aligned}
$$

The $\mathcal{R}$-matrix $\mathcal{R} \in \mathsf{S}(X) \otimes \mathsf{S}(X)$ is an invertible matrix induced by the twist

$$
\mathcal{R} = f_\alpha \bar{f}^\beta \otimes f^\alpha \bar{f}_\beta =: R^\alpha \otimes R_\alpha \ , \mathcal{R}^{-1} = \bar{R}^\alpha \otimes \bar{R}_\alpha \ ,
$$

The inverse $\mathcal{R}$-matrix controls noncommutativity of the $\vee_\star$-product and provides the representation of permutation group, e.g.,

$$
\tau_{\mathcal{R}}^\sigma(x_1 \vee_\star x_2) = (-1)^{|x_1||x_2|} \bar{R}^\alpha(x_2) \vee_\star \bar{R}_\alpha(x_1) \ ,
$$

Braided L_∞-algebra

Extend the coproduct to whole tensor algebra:

$$\Delta_\star \circ \mathrm{id}^{\vee_\star m} = \sum_{\sigma \in \mathrm{Sh}(p,m-p)} (\mathrm{id}^{\vee_\star p} \otimes \mathrm{id}^{\vee_\star (m-p)}) \circ \tau_{\mathcal{R}}^{\sigma} \ , \ p, m \geq 0 \ .$$

The compatible coderivation $D_\star = \sum_{i=0}^{\infty} b_i^\star$ is defined in terms of braided graded symmetric maps $b_i^\star$

$$b_i^\star \circ \mathrm{id}^{\vee_\star j} = \sum_{\sigma \in \mathrm{Sh}(i,j-i)} (b_i^\star \vee_\star \mathrm{id}^{\vee_\star (j-i)}) \circ \tau_{\mathcal{R}}^{\sigma} \ , \ j \geq i \ ,$$

$$b_i^\star(x_1, \dots, x_m, x_{m+1}, \dots, x_i) = (-1)^{|x_m||x_{m+1}|} b_i^\star(x_1, \dots, \bar{R}^\alpha(x_{m+1}), \bar{R}_\alpha(x_m), \dots, x_i) \ ,$$

and the condition $D_\star^2 = 0$ reproduces the deformed homotopy relations.

$\rightsquigarrow$ braided L_∞-algebra obtained in Dimitrijević Ćirić et al ‘21.

$L_\infty^\star$ vs. $L_\infty^\mathcal{F}$

As Hopf algebras $L_\infty^\star$ and $L_\infty^\mathcal{F}$ are isomorphic Aschieri et al '05, Schenkel '12
$\exists$ map $\varphi : L_\infty^\star \to L_\infty^\mathcal{F}$ such that

$$\varphi(x_1 \vee_\star x_2) = \varphi(x_1) \vee \varphi(x_2) \;,$$
$$\Delta_\star = (\varphi^{-1} \otimes \varphi^{-1}) \circ \Delta^\mathcal{F} \circ \varphi \;,$$
$$S_\star = \varphi^{-1} \circ S^\mathcal{F} \circ \varphi \;.$$

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$$\Delta_\star = (\varphi^{-1} \otimes \varphi^{-1}) \circ \Delta^\mathcal{F} \circ \varphi \; ,$$
$$S_\star = \varphi^{-1} \circ S^\mathcal{F} \circ \varphi \; .$$

On the other hand, we take $L_\infty^\star$-algebra as a module of $L_\infty^\mathcal{F}$ with an L_∞-action on an L_∞-algebra given by an L_∞-morphism Mehta, Zambon '12. Thus we obtain

$$D_\star = \varphi^{-1} D_\mathcal{F} \varphi \; .$$

DFT algebroid

Double Field Theory

Hull, Zwiebach '09, Hohm, Hull, Zwiebach '10

A proposal for a field theory defined on double, 2d-dimensional configuration space with manifest T-duality Duff '90, Tseytlin '90, Siegel '93.

It uses doubled coordinates $(x^A) = (x^a, \widetilde{x}_a)$, and fields (here only g, B) depend on both.

Global symmetry group $G = O(d,d)$ with metric

$$\eta = (\eta_{AB}) = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix} ,$$

used to raise and lower $A = 1, \ldots, 2d$ curved indices.

All objects in the theory belong to some representation of the O(d, d) duality group.

The fields in frame formalism, d-dimensional bein ${e^m}_a$ and B collect into generalized bein

$$ {\mathcal{E}^M}_A = \begin{pmatrix} {e_m}^a & {e_m}^b B_{ba} \\ & {e^m}_a \end{pmatrix} \ , \ \ {\mathcal{E}^M}_A \in G \ , $$

satisfying $\eta_{AB} {\mathcal{E}_I}^A {\mathcal{E}_J}^B = \hat{\eta}_{IJ}$ for

$$ \hat{\eta} = (\hat{\eta}_{IJ}) = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix} \ , $$

used to raise and lower $I = 1, \ldots, 2d$ flat indices.

Local gauge symmetry combines standard diffeomorphisms and gauge trafos of B, generated by an $O(d,d)$ vector $\epsilon^A = \mathcal{E}_M{}^A \lambda^M$.

The gauge algebra (of generalized Lie derivative) closes after imposing the strong constraint,

$$L_{\epsilon_1} L_{\epsilon_2} - L_{\epsilon_2} L_{\epsilon_1} = L_{[\![\epsilon_1,\epsilon_2]\!]} \ ,$$

with the bracket operation, called the C-bracket, being

$$[\![\epsilon_1, \epsilon_2]\!]^B = \epsilon_1^A \partial_A \epsilon_2^B - \tfrac{1}{2}\epsilon_1^A \partial^B \epsilon_{2A} - (\epsilon_1 \leftrightarrow \epsilon_2) \ .$$

Q: What are the properties of C-bracket?

DFT algebroid

Chatzistavrakidis, L.J, Khoo, Szabo, '18

Let $\mathcal{M}$ be a $2d$-dimensional manifold. A DFT algebroid is a quadruple $(L, [\![\cdot\,,\,\cdot]\!], \langle\cdot\,,\,\cdot\rangle_L, \rho)$, where L is vector bundle of rank $2d$ over $\mathcal{M}$ equipped with

- a skew-symmetric bracket $[\![\cdot\,,\,\cdot]\!] : \Gamma(L) \otimes \Gamma(L) \to \Gamma(L)$,
- a non-degenerate symmetric form $\langle\cdot\,,\,\cdot\rangle_L : \Gamma(L) \otimes \Gamma(L) \to C^\infty(\mathcal{M})$,
- a smooth bundle map $\rho : L \to T\mathcal{M}$,

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such that

1. $\langle \mathcal{D}f, \mathcal{D}g\rangle_L = \frac{1}{4}\,\langle \mathrm{d}f, \mathrm{d}g\rangle$
2. $[\![e_1, f\, e_2]\!] = f\,[\![e_1, e_2]\!] + \rho(e_1)f\, e_2 - \langle e_1, e_2\rangle_L\, \mathcal{D}f$
3. $\langle [\![e_3, e_1]\!] + \mathcal{D}\langle e_3, e_1\rangle_L, e_2\rangle_L + \langle e_1, [\![e_3, e_2]\!] + \mathcal{D}\langle e_3, e_2\rangle_L\rangle_L = \rho(e_3)\langle e_1, e_2\rangle_L$

for all $e_i \in \Gamma(L)$ and $f, g \in C^\infty(\mathcal{M})$, where $\mathcal{D} : C^\infty(\mathcal{M}) \to \Gamma(L)$ is the derivative defined through $\langle \mathcal{D}f, e\rangle_L = \frac{1}{2}\,\rho(e)f$.

DFT algebroid as curved L_∞

Grewcoe, LJ '20

$$\begin{array}{ccccccc} X_{-2} & \oplus & X_{-1} & \oplus & X_0 & \oplus & X_1 \\ f \in C^\infty(\mathcal{M}) & & e \in \Gamma(E) & & h \in \mathfrak{X}(\mathcal{M}) & & \eta \end{array}$$

Maps:
$b_0 = \eta$, $b_1(f) = \mathcal{D}f$, $b_1(e) = \rho(e)$, $b_2(\eta, f) = -\frac{1}{2}\eta^{-1}(df)$, ...

Homotopy relations:
$b_2(b_0, f) + b_1^2(f) = 0 \rightsquigarrow (\rho \circ \mathcal{D})f = \frac{1}{2}\eta^{-1}(df)$
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L_∞ structure $\rightsquigarrow$ cohomological vector $Q = D^*$ $\rightsquigarrow$ Q-manifold locally

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Evaluate b_i on basis of X $\rightsquigarrow$ structure constants of L_∞-algebra:

$$b_i(\tau_{\alpha_1}, ..., \tau_{\alpha_i}) = C^{\beta}_{\alpha_1 \ldots \alpha_i} \tau_\beta$$

Use to define cohomological vector Q of degree 1

$$Q = \sum_{i=0}^{\infty} \frac{1}{i!} C^{\beta}_{\alpha_1 \ldots \alpha_i} z^{\alpha_1} \cdots z^{\alpha_i} \frac{\partial}{\partial z^\beta}$$

with z^{α_i} basis of $X^\star \rightsquigarrow Q = D^*$. In DFT with ρ projection:

$$Q = \eta^{AB} \frac{\partial}{\partial s^{AB}} + \left(\delta^A_I e^I - \frac{1}{2} s^{AB} f_B \right) \frac{\partial}{\partial x^A} + \frac{1}{2} \hat{\eta}^{IJ} \delta^A_I f_A \frac{\partial}{\partial e^J},$$

Q-manifold locally as $\mathbb{R}[-1] \oplus \mathbb{R}^{2d} \oplus \mathbb{R}^{2d}[1] \oplus \mathbb{R}^{2d}[2]$ with the coordinates $\{s^{AB}, x^A, e^I, f_A\}$.

$$Q^2 = 0 \rightsquigarrow \eta^{AB} = \hat{\eta}^{IJ} \delta^A_I \delta^B_J \ ,$$

Outlook

- $Q = D^* \rightsquigarrow$ BRST operator

  Geometrically, a solution of classical master equation can be considered as a QP-manifold, i.e. a supermanifold equipped with an odd vector field Q obeying $\{Q, Q\} = 0$ and with Q-invariant odd symplectic structure. Cyclic L_∞! AKSZ '95

- In DFT algebroid no symplectic structure. Use graded contact structure to define an action?