Constructive Tensors

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Outline

Constructive

WARNING: I am biased!

Constructive field theory?

It is the branch of mathematical physics which aims at:

- defining correlation functions as holomorphic functions of the physical parameters,
- explaining the incredible agreement between perturbative computations and real experiments,
- probing universality, computing critical exponents, proving relations between thermodynamical quantities etc, in a non-perturbative way.

One should distinguish between:

- Constructive Fermionic renormalization group, well-developped in particular for statistical physics and condensed matter,
- Constructive Bosonic QFT, harder to control, still unsatisfactory despite its great successes.

Many divergences

Obstacles to non-perturbative QFT

QFT is plagued by many divergences

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Many divergences

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3. the thermodynamic limit might diverge.

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But solutions exist:

1. perturbative renormalisation deals with divergent amplitudes,
2. one can express any correlation as a uniformly convergent series (of holomorphic functions), the general term of which is indexed by a less proliferating species than graphs (*e.g.* forests),

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But solutions exist:

1. perturbative renormalisation deals with divergent amplitudes,
2. one can express any correlation as a uniformly convergent series (of holomorphic functions), the general term of which is indexed by a less proliferating species than graphs (*e.g.* forests),
3. only connected quantities have an infinite volume limit (*e.g.* the logarithm of the partition function).

Bosonic constructive field theory

A functional integral point of view

- Aim: get some control on connected quantities via the derivation of tractable formulas for the *logarithm* of the partition function (with sources).

Bosonic constructive field theory

The classical approach

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  - an *ad hoc* discretization of space,
  - a *partial* perturbative expansion,
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- Tools: cluster and Mayer expansions
  (which are both a clever application of the Taylor forest formula).

The BKAR forest formula

A botanical prerequisite

- Fix an integer $n \geqslant 2$.
- f a function of $\frac{n(n-1)}{2}$ real variables x_ℓ, sufficiently differentiable.
- K_n, complete graph on $\{1, 2, \ldots, n\}$. $\#E(K_n) = \frac{n(n-1)}{2}$

Then,

$$f(1, 1, \ldots, 1) = \sum_{\mathcal{F}} \int dw_{\mathcal{F}} \, \partial_{\mathcal{F}} f(X^{\mathcal{F}}(w_{\mathcal{F}}))$$

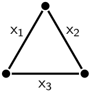
where

- the sum is over spanning forests of K_n,
- $\int dw_{\mathcal{F}} := \prod_{\ell \in E(\mathcal{F})} \int_0^1 dw_\ell$,
- $\partial_{\mathcal{F}} := \prod_{\ell \in E(\mathcal{F})} \frac{\partial}{\partial x_\ell}$,
- $X^{\mathcal{F}} = (x_\ell^{\mathcal{F}})_{\ell \in E(K_n)}$ – evaluation point of $\partial_{\mathcal{F}} f$.

The forest formula

An example

$$f(1,1,1) =$$



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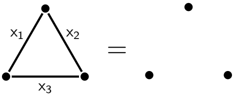

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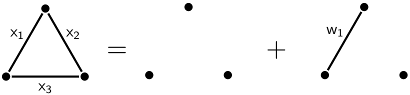

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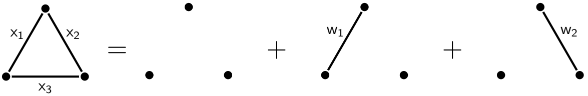
$$f(1,1,1) = f(0,0,0) + \int_0^1 \partial_{x_1} f(w_1,0,0)\, dw_1 + \int_0^1 \partial_{x_2} f(0,w_2,0)\, dw_2$$



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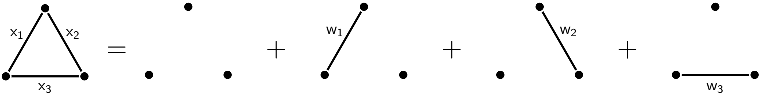

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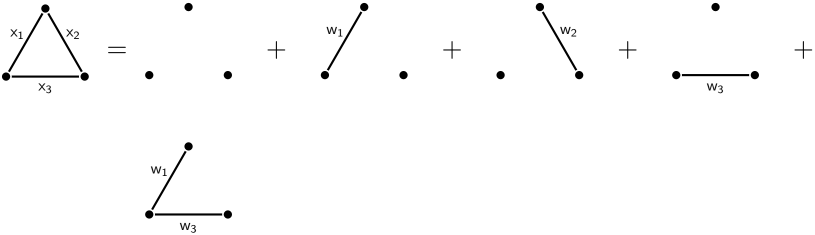

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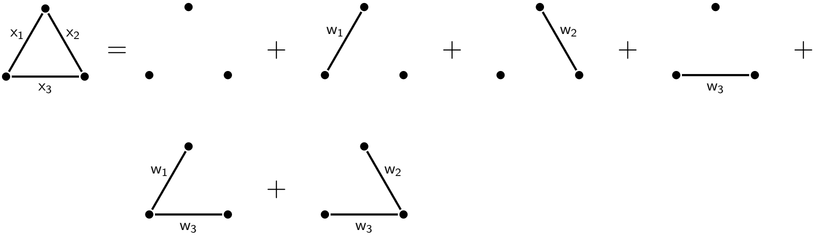

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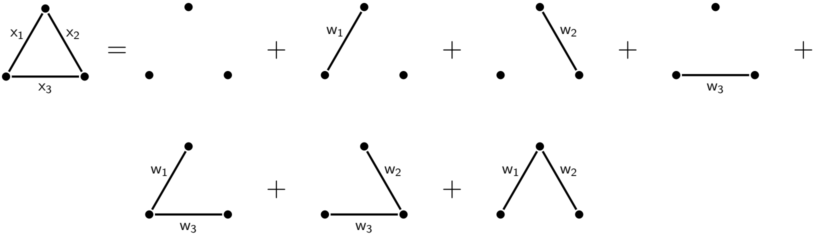

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  (which are both a clever application of the Taylor forest formula).
- But classical constructive techniques are unsuited to tensors.

Loop Vertex Expansion

Motivations

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  1. Constructive ϕ_4^{*4},
  2. Simplify Bosonic constructive theory.

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1. Classical constructive techniques unsuited to matrices/tensors.
   They rely on:
   - locality of the interaction,
   - strong spatial decay of the propagator.

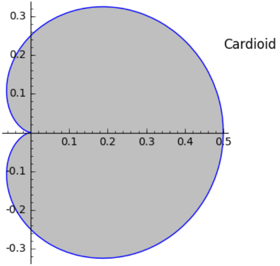
Loop Vertex Expansion

Analyticity of the free energy of ϕ_0^4

$$Z(\lambda) = \int_{\mathbb{R}} e^{-\frac{\lambda}{2}\phi^4}\, d\mu(\phi), \quad d\mu(\phi) = \frac{d\phi}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}$$

Theorem

$\log Z$ *is analytic in the cardioid domain* $\left\{\lambda \in \mathbb{C} : |\lambda| < \frac{1}{2}\cos^2(\frac{1}{2}\arg\lambda)\right\}$.



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Proof. LVE is done in 3 steps:

1. Intermediate field representation,
2. Replication of fields,
3. Forest formula.

Loop Vertex Expansion

Analyticity of the free energy of ϕ^4_0

1. Intermediate field representation: $e^{-\frac{\lambda}{2}\phi^4} = \int_{\mathbb{R}} e^{\imath\sigma\phi^2\sqrt{\lambda}}\, d\mu(\sigma)$

$$
\begin{aligned}
Z(\lambda) &= \int_{\mathbb{R}} e^{-\frac{\lambda}{2}\phi^4}\, d\mu(\phi) \\
&= \int_{\mathbb{R}} e^{V(\sigma)}\, d\mu(\sigma), \quad V(\sigma) = -\tfrac{1}{2}\log(1 - \imath\sigma\sqrt{\lambda}).
\end{aligned}
$$

Loop Vertex Expansion

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2. Replication of fields:

$$Z(\lambda) = \sum_{n\geqslant 0} \frac{1}{n!} \int_{\mathbb{R}} V(\sigma)^n\, d\mu(\sigma) = \sum_{n\geqslant 0} \frac{1}{n!} \int_{\mathbb{R}^n} \Big(\prod_{i=1}^{n} V(\sigma_i)\Big)\, d\mu_{\mathbb{1}_n}(\vec{\sigma}).$$

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Each “order” consists in the resummation
of infinitely many Feynman graphs.

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3. Forest formula:

$$Z = \sum_{n\geqslant 0} \frac{1}{n!} \sum_{\mathcal{F}\subset K_n} \int dw_{\mathcal{F}} \int \Big[\prod_{\ell\in E(\mathcal{F})} \frac{\partial}{\partial\sigma_{i(\ell)}} \frac{\partial}{\partial\sigma_{j(\ell)}} \prod_{i=1}^n V(\sigma_i)\Big]\, d\mu_{C^{\mathcal{F}}(w)}(\vec{\sigma})$$

$$\log Z = \sum_{n\geqslant 1} \frac{1}{n!} \sum_{\mathcal{T}\subset K_n} \int dw_{\mathcal{T}} \int \Big[\prod_{\ell\in E(\mathcal{T})} \frac{\partial}{\partial\sigma_{i(\ell)}} \frac{\partial}{\partial\sigma_{j(\ell)}} \prod_{i=1}^n V(\sigma_i)\Big]\, d\mu_{C^{\mathcal{T}}(w)}(\vec{\sigma}).$$

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$$= \frac{1}{2} \sum_{n\geqslant 1} \frac{(-\lambda/2)^{n-1}}{n!} \sum_{\mathcal{T}\subset K_n} \int dw_{\mathcal{T}} \int \Big[\prod_{i=1}^{n} \frac{(d_i-1)!}{(1-\imath\sigma_i\sqrt{\lambda})^{d_i}} \Big] \, d\mu_{C^{\mathcal{T}}(w)}(\vec{\sigma}).$$

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Using $\left|1-\imath\sigma\sqrt{\lambda}\right| \geqslant \cos(\frac{1}{2}\arg\lambda)$, we get

$$|\log Z| \leqslant \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n!} \Big(\frac{|\lambda|}{2\cos^2(\frac{1}{2}\arg\lambda)} \Big)^{n-1} \sum_{\mathcal{T}\subset K_n} \prod_{i=1}^{n} (d_i-1)!$$

$$\leqslant 2 \sum_{n=1}^{\infty} \Big(\frac{2|\lambda|}{\cos^2(\frac{1}{2}\arg\lambda)} \Big)^{n-1}$$

which is convergent for all $\lambda\in\mathbb{C}$ such that $|\lambda| < \frac{1}{2}\cos^2(\frac{1}{2}\arg\lambda)$. □

Tensors

Constructive

Tensors

- Why?
- How?

Constructive Tensors

Random surfaces

- $2D$ quantum gravity and matrix models:
  - Matrix models provide a theory of random discrete surfaces weighted by a discretized Einstein-Hilbert action.
  - Evidence for matrices being the right discretization of $2D$ quantum gravity.

- In the last 10 years, much progress from probabilists:
  - Random metric surfaces (*e.g.* Brownian map).
  - Universal limit of large planar maps.
  - Liouville quantum gravity sphere = Brownian map.

Why tensor fields?

1. Generalize matrix models to higher dimensions
   - w.r.t. their symmetry properties,
   - provide a theory of random spaces.

2. Define a canonical way of summing over spaces

3. Implement a geometrogenesis scenario
   - spacetime from scratch,
   - background independent.

Invariant actions

Symmetry

Consider $T, \overline{T} : \mathbb{Z}^D \to \mathbb{C}$, complex rank D tensors with *no symmetry*.

Invariant actions

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- Matrix models: invariant under (at most) two copies of $U(N)$.
  Tensor models (rank D): invariant under D copies of $U(N)$.

$$T_{n_1 n_2 \cdots n_D} \longrightarrow \sum_m U^{(1)}_{n_1 m_1} U^{(2)}_{n_2 m_2} \cdots U^{(\mathrm{D})}_{n_D m_D} T_{m_1 m_2 \cdots m_D}$$

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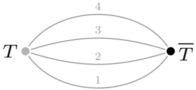

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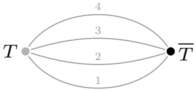

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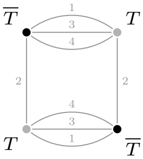

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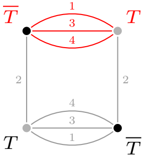

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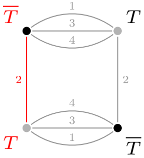

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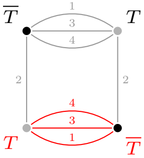

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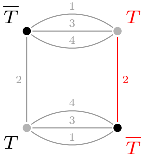

$$\sum_{m_i, n_j} T_{m_1 m_2 m_3 m_4} \overline{T}_{m_1 n_2 m_3 m_4} T_{n_1 n_2 n_3 n_4} \overline{T}_{n_1 m_2 n_3 n_4}$$

Invariant actions

Symmetry

Consider $T, \overline{T} : \mathbb{Z}^D \to \mathbb{C}$, complex rank D tensors with *no symmetry*.

- Matrix models: invariant under (at most) two copies of $U(N)$.
  Tensor models (rank D): invariant under D copies of $U(N)$.

$$T_{n_1 n_2 \cdots n_D} \longrightarrow \sum_m U^{(1)}_{n_1 m_1} U^{(2)}_{n_2 m_2} \cdots U^{(D)}_{n_D m_D} T_{m_1 m_2 \cdots m_D}$$

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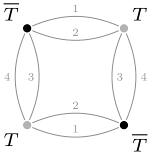

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Invariant actions

Feynman graphs

- Action of a tensor model

$$S(T,\overline{T}) = T\cdot\overline{T} + \sum_{\mathcal{B}\in\mathfrak{I}} g_{\mathcal{B}}\,\mathrm{Tr}_{\mathcal{B}}[T,\overline{T}],$$

$$\mathfrak{I}\subset\{D\text{-coloured graphs of order}\geqslant 4\}$$ interaction vertices

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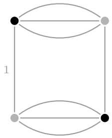

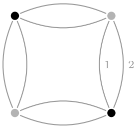

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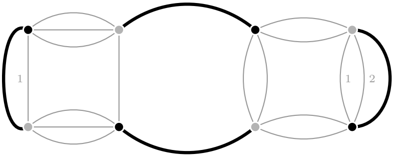

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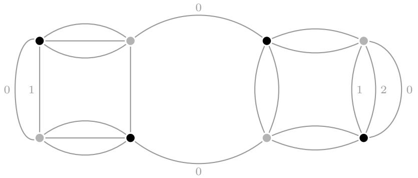

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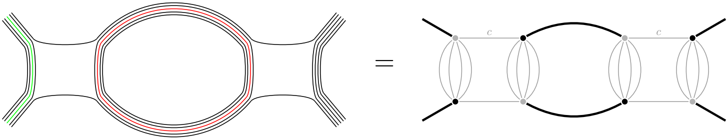

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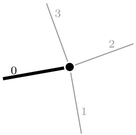
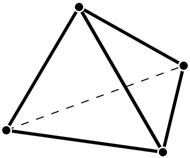
- Feynman graphs $=$ $(D+1)$-coloured graphs $=$ D-Triang. spaces

vertex



D-simplex



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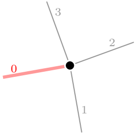
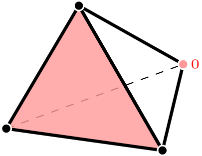
half-edge



$(D-1)$-face



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interaction vertices

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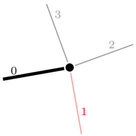

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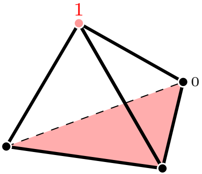

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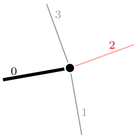

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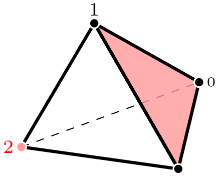

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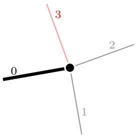

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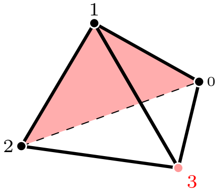

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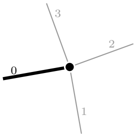

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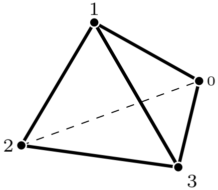

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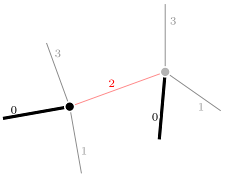
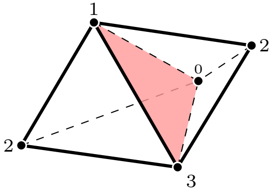


gluing



Random geometry

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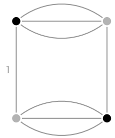
- Let us consider a sequence of random $(D+1)$-coloured graphs, *e.g.* $(\mathcal{Q}_n^{(\mathrm{N})})_{n\geqslant 0}$ where $\mathcal{Q}_n^{(\mathrm{N})}$ is a random quartic melonic graph of order n such that for any fixed quartic melonic graph G, $\mathbb{P}\left[\mathcal{Q}_n = G\right] \propto N^{D-\frac{2}{(D-1)!}\omega(G)}$.



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- **The large N limit** (a.k.a. $N \to \infty$): only Gurau degree 0 graphs survive and, if properly renormalised, as $n \to \infty$, $|\mathcal{Q}_n^{(\mathrm{N})}| \to$ continuous random *tree* (for any $D \geqslant 3$). [Gurau & Ryan, 2013]

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- $\mathbf{1 < N < \infty}$: hard, no result!

Constructive Tensors

The T_4^4 field theory

The hardest theory we can handle (joint work with V. Rivasseau)

- Tensor fields:

$$T : \mathbb{Z}^4 \to \mathbb{C}, \qquad T_{\boldsymbol{n}}, \overline{T}_{\overline{\boldsymbol{n}}} \text{ with } \boldsymbol{n}, \overline{\boldsymbol{n}} \in \mathbb{Z}^4.$$

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- Free action:

$$d\mu_C(T, \overline{T}) = \left(\prod_{\boldsymbol{n}, \overline{\boldsymbol{n}}} \frac{dT_n d\overline{T}_{\overline{n}}}{2i\pi} \right) \det(C^{-1}) \ e^{-\sum_{n,\bar{n}} T_n C^{-1}_{n\bar{n}} \overline{T}_{\bar{n}}},$$

$$C_{\boldsymbol{n},\overline{\boldsymbol{n}}} = \frac{(\mathbf{1}_{\leqslant j_{\max}})_{\boldsymbol{n}\overline{\boldsymbol{n}}}}{\boldsymbol{n}^2 + 1} \, \delta_{\boldsymbol{n},\overline{\boldsymbol{n}}}, \quad \boldsymbol{n}^2 := n_1^2 + n_2^2 + n_3^2 + n_4^2,$$

$$\mathbf{1}_{\leqslant j_{\max}} = 1_{\boldsymbol{n}^2 + 1 \leqslant M^{2j_{\max}}} \, \delta_{\boldsymbol{n},\overline{\boldsymbol{n}}}.$$

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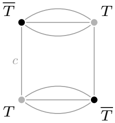
- Interactions:

$$V(T, \overline{T}) = \frac{g}{2} \sum_{c=1}^{4} V_c(T, \overline{T}), \qquad V_c(T, \overline{T}) =$$



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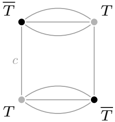

- Formal partition function:

$$Z_0(g) = \int e^{-\frac{g}{2} \sum_c V_c(T,\overline{T})} \, d\mu_C(T,\overline{T}).$$

The T_4^4 field theory

Perturbative renormalisation

Proposition

T_4^4 is renormalisable to all orders of perturbation.

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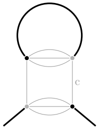
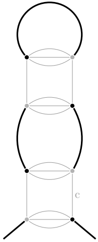


$\mathcal{M}_1^c$



$\mathcal{M}_2^c$

Proposition

T_4^4 is renormalisable to all orders of perturbation.

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- 2 divergent 2-point graphs: $\mathcal{M}_1^c, \mathcal{M}_2^c$
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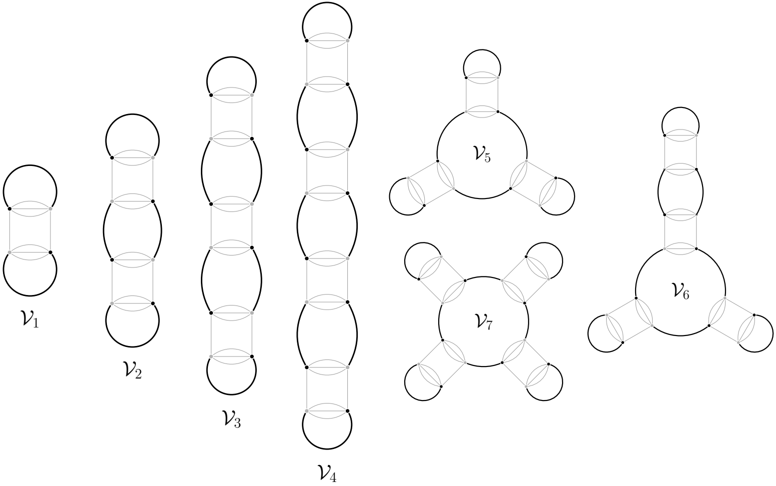

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Perturbative renormalisation

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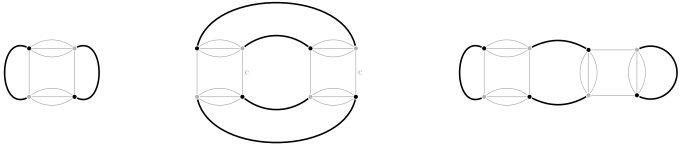
- 7 divergent melonic vacuum graphs: $\mathcal{V}_i, i = 1, 2, \ldots, 7$
- 3 divergent non melonic vacuum graphs:



$\mathfrak{N}_1$ $\mathfrak{N}_2$ $\mathfrak{N}_3$

The T^4_4 field theory

Analyticity

- Renormalised partition function:

$$Z_{j_{\max}}(g) = \mathcal{N} \int e^{-\frac{g}{2}\sum_c V_c(T,\overline{T}) + T\cdot\overline{T}\left(\sum_{G\in\mathcal{M}} \frac{(-g)^{|G|}}{S_G}\,\delta_G\right)}\, d\mu_C(T,\overline{T}),$$

$$\mathcal{M} = \{\mathcal{M}_1^c, \mathcal{M}_2^c,\ c = 1,2,3,4\}\,,$$

$\log\mathcal{N}$ = (finite) sum of the counterterms of the divergent vacuum connected graphs.

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Theorem (Rivasseau, V.-T. 2017)

There exists $\rho > 0$ such that $\lim_{j_{max}\to\infty} \log Z_{j_{max}}(g)$ is analytic in the cardioid domain defined by $|\arg g| < \pi$ and $|g| < \rho\cos^2(\frac{1}{2}\arg g)$.

The general strategy

0. Renormalised partition function:

$$Z_{j_{\max}}(g) = \mathcal{N} \int e^{-\frac{g}{2}\sum_c V_c(T,\overline{T}) + T\cdot\overline{T}\left(\sum_{G\in\mathcal{M}} \frac{(-g)^{|G|}}{S_G}\,\delta_G\right)}\, d\mu_C(T,\overline{T}).$$

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1. Intermediate field representation: $\sigma_c \in \mathrm{Herm}_{M^{j_{\max}}}$, $c = 1,2,3,4$

$$Z_{j_{\max}}(g) = \mathcal{N} e^{\delta_t} \int e^{-\operatorname{Tr}\log(\mathbb{I}-\Sigma) - \imath\lambda \sum_c \delta^c_m \operatorname{Tr}_c \sigma_c}\, d\nu_{\mathbb{I}}(\vec{\sigma})$$

$$\begin{aligned}
\lambda &= g^{1/2} \\
\Sigma &= \imath\lambda C^{1/2}\sigma C^{1/2} \\
\sigma &= \sigma_1 \otimes \mathbb{I}_2 \otimes \mathbb{I}_3 \otimes \mathbb{I}_4 + \mathbb{I}_1 \otimes \sigma_2 \otimes \mathbb{I}_3 \otimes \mathbb{I}_4 + \cdots
\end{aligned}$$

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$$Z_{j_{\max}}(g) = \mathcal{N} \int e^{-\frac{g}{2}\sum_c V_c(T,\overline{T}) + T\cdot\overline{T}\left(\sum_{G\in\mathcal{M}} \frac{(-g)^{|G|}}{S_G}\,\delta_G\right)} d\mu_C(T,\overline{T}).$$

1. Intermediate field representation: $\sigma_c \in \mathrm{Herm}_{M^{j_{\max}}}$

$$Z_{j_{\max}}(g) = \mathcal{N} e^{\delta_t} \int e^{-\operatorname{Tr}\log(\mathbb{I}-\Sigma) - \imath\lambda \sum_c \delta_m^c \operatorname{Tr}_c \sigma_c}\, d\nu_{\mathbb{I}}(\vec{\sigma}).$$

2. Renormalised *action*:

$$Z_{j_{\max}}(g) = \int e^{-\operatorname{Tr}\log_3(\mathbb{I}-U) - \operatorname{Tr}(D_1\Sigma^2) - \frac{\lambda^2}{2} :\vec{\sigma}\cdot Q\vec{\sigma}:}\, d\nu_{\mathbb{I}}(\vec{\sigma})$$

$$U = \Sigma + D_1 + D_2$$

$$D_1 = -\lambda^2 C^{1/2} A^{\mathrm{r}}_{\mathcal{M}_1} C^{1/2}, \qquad D_2 = \lambda^4 C^{1/2} A^{\mathrm{r}}_{\mathcal{M}_2} C^{1/2}$$

$$\log_3(\mathbb{I}-U) = \log(\mathbb{I}-U) + U + \frac{U^2}{2}$$

The general strategy

3. Multiscale decomposition:

$$C_{\leqslant j} := \delta_{\boldsymbol{n},\bar{\boldsymbol{n}}} \frac{(\mathbf{1}_{\leqslant j})_{\boldsymbol{n}\bar{\boldsymbol{n}}}}{\boldsymbol{n}^2 + 1}$$

$$V_{\leqslant j} = \operatorname{Tr} \log_3[\mathbb{I} - U_{\leqslant j}] + \operatorname{Tr}[D_{1,\leqslant j} \Sigma^2_{\leqslant j}] + \frac{\lambda^2}{2} {:}\vec{\sigma} \cdot Q_{\leqslant j} \vec{\sigma}{:}$$

$$V_{\leqslant j_{\max}} = \sum_{j=1}^{j_{\max}} (V_{\leqslant j} - V_{\leqslant j-1}) =: \sum_{j=1}^{j_{\max}} V_j$$

$$Z_{j_{\max}}(g) = \int \prod_j e^{-V_j} \, d\nu_{\mathbb{I}}(\vec{\sigma}) = \int e^{-\sum_j \bar{\chi}_j W_j(\sigma) \chi_j} \, d\nu_{\mathbb{I}}(\vec{\sigma}) d\mu(\bar{\chi}, \chi)$$

$$W_j = e^{-V_j} - 1$$

The general strategy

0. Renormalised partition function:

$$Z_{j_{\max}}(g) = \mathcal{N} \int e^{-\frac{g}{2}\sum_c V_c(T,\overline{T}) + T\cdot\overline{T}\left(\sum_{G\in\mathcal{M}} \frac{(-g)^{|G|}}{S_G}\,\delta_G\right)}\, d\mu_C(T,\overline{T}).$$

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The general strategy

4. Multiscale Loop Vertex Expansion: [Gurau, Rivasseau 2014]
   - 2 forest formulas on top of each other (2-jungle formula)
   - First, a Bosonic forest then a Fermionic one

The general strategy

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The general strategy

4. Multiscale Loop Vertex Expansion: [Gurau, Rivasseau 2014]
   - 2 forest formulas on top of each other (2-jungle formula)
   - First, a Bosonic forest then a Fermionic one

$$\log Z_{j_{\max}}(g) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{J} \text{ tree}} \sum_{j_1=1}^{j_{\max}} \cdots \sum_{j_n=1}^{j_{\max}} \int d\boldsymbol{w}_{\mathcal{J}} \int d\nu_{\mathcal{J}} \, \partial_{\mathcal{J}} \Big[\prod_{\mathcal{B}} \prod_{a \in \mathcal{B}} \big(-\bar{\chi}^{\mathcal{B}}_{j_a} W_{j_a}(\vec{\sigma}^a) \chi^{\mathcal{B}}_{j_a}\big) \Big]$$

- $\boldsymbol{w}_{\mathcal{J}}$ = weakening parameters w_ℓ, $\ell \in E(\mathcal{J})$
- $\nu_{\mathcal{J}}$ = interpolated Gaussian Bosonic and Fermionic measures
- $\partial_{\mathcal{J}}$ = derivatives with respect to the σ-, χ- and $\bar{\chi}$-fields

The general strategy

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$$= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{J} \text{ tree}} \sum_{j_1=1}^{j_{\max}} \cdots \sum_{j_n=1}^{j_{\max}} \int d\mathbf{w}_{\mathcal{J}} \prod_{\mathcal{B}} \Big(\prod_{\substack{a,b \in \mathcal{B} \\ a \neq b}} (1 - \delta_{j_a j_b}) \Big) I_{\mathcal{B}}$$

$$I_{\mathcal{B}} = \int \partial_{\mathcal{B}} \prod_{a \in \mathcal{B}} W_{j_a}(\vec{\sigma}^a)\, d\nu_{\mathcal{B}} = \sum_{\mathrm{G}} \int \Big(\prod_{a \in \mathcal{B}} e^{-V_{j_a}(\vec{\sigma}^a)} \Big) A_{\mathrm{G}}(\vec{\sigma})\, d\nu_{\mathcal{B}}(\vec{\sigma})$$

- Graphs G are plane forests.

The general strategy

Bosonic bounds

$$I_{\mathcal{B}} = \sum_{\mathrm{G}} \int \Big(\prod_{a \in \mathcal{B}} e^{-V_{j_a}(\vec{\sigma}^a)} \Big) A_{\mathrm{G}}(\vec{\sigma})\, d\nu_{\mathcal{B}}(\vec{\sigma})$$

$$|I_{\mathcal{B}}| \leqslant \underbrace{\Big(\int \prod_{a \in \mathcal{B}} e^{2|V_{j_a}(\vec{\sigma}^a)|}\, d\nu_{\mathcal{B}} \Big)^{1/2}}_{I_{\mathcal{B}}^{NP},\ \text{non-perturbative}} \sum_{\mathrm{G}} \underbrace{\Big(\int \big| A_{\mathrm{G}}(\vec{\sigma}) \big|^2\, d\nu_{\mathcal{B}} \Big)^{1/2}}_{I_{\mathcal{B}}^{P},\ \text{perturbative}}$$

The general strategy

Bosonic bounds

$$|I_{\mathcal{B}}| \leqslant \underbrace{\Big(\int \prod_{a\in\mathcal{B}} e^{2|V_{j_a}(\vec{\sigma}^a)|}\, d\nu_{\mathcal{B}}\Big)^{1/2}}_{I_{\mathcal{B}}^{NP},\ \text{non-perturbative}} \sum_{\mathrm{G}} \underbrace{\Big(\int \big|A_{\mathrm{G}}(\vec{\sigma})\big|^2\, d\nu_{\mathcal{B}}\Big)^{1/2}}_{I_{\mathcal{B}}^{P},\ \text{perturbative}}$$

5. Non-perturbative bound

"where we use an old idea of Glimm and Jaffe"

Theorem

For ρ small enough and for any value of the w interpolating parameters,

$$I_{\mathcal{B}}^{NP} = \int \prod_{a\in\mathcal{B}} e^{2\left|V_{j_a}(\vec{\sigma}^a)\right|}\, d\nu_{\mathcal{B}} \leqslant O(1)^{|\mathcal{B}|}.$$

The general strategy

Bosonic bounds

$$|I_{\mathcal{B}}| \leqslant \underbrace{\Big(\int \prod_{a\in\mathcal{B}} e^{2|V_{j_a}(\vec{\sigma}^a)|}\, d\nu_{\mathcal{B}}\Big)^{1/2}}_{I_{\mathcal{B}}^{NP},\ \text{non-perturbative}} \sum_{\mathrm{G}} \underbrace{\Big(\int \big|A_{\mathrm{G}}(\vec{\sigma})\big|^2\, d\nu_{\mathcal{B}}\Big)^{1/2}}_{I_{\mathcal{B}}^{P},\ \text{perturbative}}$$

6. Perturbative bound

"where we heavily rely on the fact that trees in the σ-representation correspond to stranded graphs with all their faces"

Theorem

Let $\mathcal{B}$ be a Bosonic block. Then there exists $K \in \mathbb{R}_+^*$ such that

$$I_{\mathcal{B}}^{P}(\mathrm{G}) \leqslant K^{|\mathcal{B}|-1}\big((|\mathcal{B}|-1)!\big)^{37/2} \rho^{e(\mathrm{G})} \prod_{a\in\mathcal{B}} M^{-\frac{1}{48}j_a}.$$

The general strategy

0. Renormalised partition function
1. Intermediate field representation
2. Renormalised action
3. Multiscale decomposition
4. Multiscale Loop Vertex Expansion
5. Non-perturbative Bosonic bound
6. Perturbative Bosonic bound

The general strategy

0. Renormalised partition function
1. Intermediate field representation
2. Renormalised action
3. Multiscale decomposition
4. Multiscale Loop Vertex Expansion
5. Non-perturbative Bosonic bound
6. Perturbative Bosonic bound
7. Put everything together and cross your fingers... $\square$

Conclusion and perspectives

- Tensor field theory provides a combinatorial theory of random D-spaces.
- In the last 10^+ years, a lot of results with tensors: $\frac{1}{N}$-expansion, perturbative and constructive renormalisation, continuum limit of the dominant triangulations, double scaling limit, uniform random complexes etc.
- Regarding T_4^4, one could also prove Borel summability and analyticity of the connected correlation functions.

Conclusion and perspectives

- Tensor field theory provides a combinatorial theory of random D-spaces.
- In the last 10^+ years, a lot of results with tensors: $\frac{1}{N}$-expansion, perturbative and constructive renormalisation, continuum limit of the dominant triangulations, double scaling limit, uniform random complexes etc.
- Regarding T^4_4, one could also prove Borel summability and analyticity of the connected correlation functions.

- T^4_5 (just renormalisable, asymptotically free)
- New Loop Vertex Representation [Rivasseau 2017]
- Simplify Bosonic constructive theory?

Thank you for your attention

Non-perturbative bounds

Theorem

For ρ small enough and for any value of the w interpolating parameters,

$$I^{NP}_{\mathcal{B}} = \int \prod_{a\in\mathcal{B}} e^{2\left|V_{j_a}(\vec{\sigma}^a)\right|} \, d\nu_{\mathcal{B}} \leqslant O(1)^{|\mathcal{B}|} e^{O(1)\rho^{3/2}|\mathcal{B}|}.$$

Proof.

1. Expand each node:

$$e^{2|V_{j_a}|} = \sum_{k=0}^{p_a} \frac{(2|V_{j_a}|)^k}{k!} + \int_0^1 dt_{j_a} (1-t_{j_a})^{p_a} \frac{(2|V_{j_a}|)^{p_a+1}}{p_a!} e^{2t_{j_a}|V_{j_a}|}.$$

2. Crude non-perturbative bound: (Quadratic bound)

$$\int \prod_{a\in\mathcal{B}} e^{2\left|V_{j_a}(\vec{\sigma}^a)\right|} \, d\nu_{\mathcal{B}} \leqslant K^{|\mathcal{B}|} e^{K'\rho M^{j_1}}.$$

3. Power counting (via quartic bound) beats both combinatorics and the crude non-perturbative bound. □

Perturbative bound

Theorem

Let $\mathcal{B}$ be a Bosonic block. Then there exists $K \in \mathbb{R}_+^*$ such that

$$I_{\mathcal{B}}^P(\mathrm{G}) = \int \left|A_{\mathrm{G}}(\vec{\sigma})\right|^2 d\nu_{\mathcal{B}} \leqslant K^{|\mathcal{B}|-1}\left((|\mathcal{B}|-1)!\right)^{37/2} \rho^{e(\mathrm{G})} \prod_{a\in\mathcal{B}} M^{-\frac{1}{48}j_a}.$$

- $A_{\mathrm{G}}(\vec{\sigma})$ depends on σ (essentially) through resolvents.
- If not for resolvents, A_{G} would be the amplitude of a convergent graph.
- Remove resolvents with iterated Cauchy-Schwarz estimates.