Lie superalgebra cohomology and (perhaps) new branes

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Outline

- Motivations: (super)branes
- Forms on Supermanifolds (Berezin, Bernstein, Leites, Manin, Penkov, Witten, etc.)
- Lie Algebra Cohomology (Chevalley-Eilenberg, Koszul, Hochschild-Serre, etc.)
- Lie Superalgebra Cohomology: superforms (Kac, Fuks, etc.)
- Lie Superalgebra Cohomology: integral and pseudoforms

Green-Schwarz Superstrings and κ-symmetry

GS formulation of superstrings is constructed to have manifest target supersymmetry. Given a manifold $(\Sigma, \mathcal{O}_\Sigma)$ and a supermanifold $(\mathcal{M}, \mathcal{O}_\mathcal{M})$:

$$\phi : \Sigma \longrightarrow \mathcal{M} \ ,$$

Σ is the $(2-d)$ world-sheet spanned by the string, $\mathcal{M}$ is the target (super)manifold in which the string lives, with dimension $\dim \mathcal{M} = (m|n)$. On functions:

$$\begin{array}{rcl} \phi^* : \mathcal{O}_\mathcal{M} & \longrightarrow & \mathcal{O}_\Sigma \\ f\,(X,\theta) & \mapsto & \phi^* f\,(X(\sigma), \Theta(\sigma)) \ , \end{array}$$

and analogously on forms. Let us fix $\mathcal{M}$ to be 10-d, $N=2$ super-Minkowski (hence $\dim \mathcal{M} = (10|32)$, type IIB), a basis for the supersymmetry generators reads

$$Q_{\alpha A} = \partial_{\alpha A} - i\,(\theta \Gamma^\mu)^\alpha_A\,\partial_\mu \ , \ \alpha = 1, \dots, 32 \ , \ \mu = 1, \dots, 10 \ , \ A = 1, 2 \ .$$

A basis of manifestly supersymmetric (1|0)-forms reads

$$V^\mu = dX^\mu - i\bar{\theta}^{\alpha A}\Gamma^\mu_{\alpha\beta} d\theta^{\beta A} \quad , \quad \psi^{\alpha A} = d\theta^{\alpha A} \quad , \quad \mathcal{L}_{Q_\alpha} V^\mu = 0 = \mathcal{L}_{Q_\alpha} \psi^\beta \ ,$$

and their pull-back reads

$$\begin{aligned} \phi^*\left(V^\mu\right) = \left(V^\mu\right)^* &= \left(\partial_i X^\mu - i\bar{\theta}^{\alpha A}\Gamma^\mu_{\alpha\beta}\partial_i\theta^{\beta A}\right) d\sigma^i = \Pi^\mu_i d\sigma^i \ , \\ \phi^*\left(\psi^\alpha\right) \equiv \left(\psi^\alpha\right)^* &= \partial_i\theta^{\alpha A} d\sigma^i \ . \end{aligned}$$

We can use them to construct the superstring action as a *sigma model* (Polyakov action):

$$S = -\frac{1}{2\pi}\int_\Sigma \left[d^2\sigma\right] \Pi^\mu_i \Pi^\nu_j \eta_{\mu\nu}\sqrt{h}h^{ij} \ ,$$

where h_{ij} is the world-sheet metric. By putting on-shell the world-sheet metric $h_{ij} = \Pi^\mu_i \Pi^\nu_j \eta_{\mu\nu}$ the action is written *à la* Nambu-Goto:

$$S = -\frac{1}{\pi}\int_\Sigma \sqrt{\det\left(\Pi^\mu_i \Pi^\nu_j \eta_{\mu\nu}\right)} \ ,$$

The (on-shell) degrees of freedom do not match: 8 bosonic vs 16 fermionic. But, we can add a second term to the action

$$S' = c \int_\Sigma \left[-i\epsilon^{ij}\partial_i X^\mu \left(\bar{\theta}^{\alpha 1}\Gamma_{\mu\alpha\beta}\partial_j\theta^{\beta 1} - \bar{\theta}^{\alpha 2}\Gamma_{\mu\alpha\beta}\partial_j\theta^{\beta 2} \right) + \right.$$

$$\left. + \epsilon^{ij}\bar{\theta}^{\alpha 1}\Gamma^\mu_{\alpha\beta}\partial_i\theta^{\beta 1}\bar{\theta}^{\gamma 2}\Gamma_{\mu\gamma\delta}\partial_j\theta^{\delta 2} \right] d\sigma^1 \wedge d\sigma^2 \ ,$$

so that $S + S'$ has a new, fermionic gauge symmetry, the κ *symmetry*:

$$\begin{aligned} \delta_\kappa \theta^{\alpha 1} &= 2i\Gamma^{\alpha\beta}_\mu \Pi^\mu_i P^{ij}_- K^1_{j\beta} \ , \\ \delta_\kappa \theta^{\alpha 2} &= 2i\Gamma^{\alpha\beta}_\mu \Pi^\mu_i P^{ij}_+ K^2_{j\beta} \ , \\ \delta_\kappa X^\mu &= i\bar{\theta}^{\alpha A}\gamma^\mu_{\alpha\beta}\delta_\kappa\theta^{\beta A} \ , \end{aligned}$$

where $P_\pm = \frac{1}{2}\left(h^{ij} \pm \frac{\epsilon^{ij}}{\sqrt{h}} \right)$ are projectors $\implies$ gauge away half of the fermionic degrees of freedom.

Problems:

- difficult to check ($N = 2$) supersymmetry of S';
- "far from obvious term", i.e., difficult to guess in different dimensions/backgrounds.

Solution: *Lie algebra cohomology*.

$$\text{Chevalley-Eilenberg : Lie algebra cohomology classes} \leftrightarrow \text{invariants} \ .$$

In our example, we have $\omega^{(3|0)} \in H^{(3|0)}(\mathfrak{g})$. On the other hand, w.r.t. de Rham cohomology, $\omega^{(3|0)} = d\eta^{(2|0)}$, so that

$$S' = \int_\Lambda \omega^{(3|0)} = \int_\Lambda d\eta^{(2|0)} = \int_\Sigma \eta^{(2|0)} \ , \ \Sigma = \partial\Lambda \ .$$

Supersymmetry, reparametrisation invariance and Poincaré are guaranteed. The total superstring action reads as a sigma model + Wess-Zumino term.

The whole problem is geometrised:

- the action is written in term of forms

$$S + S' = -\frac{1}{2\pi}\int_\Sigma (V^\mu)^* \wedge \star (V^\nu)^* \eta_{\mu\nu} + c\int_\Lambda \left(V^\mu \bar{\psi}^\alpha \Gamma_{\mu\alpha\beta}\psi^\alpha\right)^* \ ,$$

"$\star$" being the Hodge dual operator w.r.t. the world-sheet metric h_{ij}.

- the susy algebra is encoded into the Maurer-Cartan forms V^μ and ψ^α

$$dV^\mu = -i\bar{\psi}^\alpha \Gamma^\mu_{\alpha\beta}\psi^\beta \ , \ d\psi^\alpha = 0 \ ;$$

- infinitesimal transformations are Lie derivatives

$$susy : \ \delta_\epsilon = \mathcal{L}_\epsilon = d\iota_\epsilon + \iota_\epsilon d \ , \ \kappa : \ \delta_\kappa = \mathcal{L}_\kappa = d\iota_\kappa + \iota_\kappa d \ ,$$

where $\epsilon = \epsilon^\alpha Q_\alpha$ and $\kappa = \kappa^\alpha D_\alpha$, where $D_\alpha = \partial_\alpha - i\left(\theta\Gamma^\mu\right)^\alpha \partial_\mu$ is the superderivative. In particular,

$$\delta_\kappa S' = c\int_\Lambda \left[\delta_\kappa \left(V^\mu \bar{\psi}^\alpha \Gamma_{\mu\alpha\beta}\psi^\alpha\right)\right]^* = c\int_\Lambda d\left[\iota_\kappa \left(V^\mu \bar{\psi}^\alpha \Gamma_{\mu\alpha\beta}\psi^\alpha\right)\right]^* =$$

$$= c\int_\Sigma \left[\iota_\kappa \left(V^\mu \bar{\psi}^\alpha \Gamma_{\mu\alpha\beta}\psi^\alpha\right)\right]^* = -2c\int_\Sigma \left(V^\mu \bar{\kappa}^\alpha \Gamma_{\mu\alpha\beta}\psi^\alpha\right)^* \ .$$

Branes, Supergravity, FDA's

- Brane Scan (Achucarro-Evans-Townsend-Wiltshire, Green-Schwarz-Witten, Duff, Baez-Huerta etc.): the existence of terms that allow to implement κ symmetry for strings is guaranteed only in $d = 3, 4, 6, 10 \iff$ Fierz identities

$$\bar{\psi}\Gamma^{\mu}\psi\bar{\psi}\Gamma_{\mu}\psi = 0 \ ;$$

  Higher Branes: 2-brane in $d = 4, 6, 7, 11 \iff$ Fierz identities

$$\bar{\psi}\Gamma^{\mu\nu}\psi\bar{\psi}\Gamma_{\mu}\psi = 0 \ ;$$

- κ transformation for p-branes:

$$\delta_\kappa S = \int_\Sigma \Pi_i^\mu \Gamma_{\mu\alpha\beta}\epsilon^{ij}\left(\mathbb{I} - \gamma\right)_\gamma^\alpha \kappa^\gamma \partial_j \theta^\beta \ ,$$

$$\gamma_\beta^\alpha = \frac{(-1)^{(p-1)(p+2)/4}}{(p-1)!\sqrt{h}}\epsilon^{i_1 \ldots i_{p+1}}\Pi_{i_1}^{\mu_1} \ldots \Pi_{i_{p+1}}^{\mu_{p+1}}\left(\Gamma_{\mu_1 \ldots \mu_{p+1}}\right)_\beta^\alpha \ ;$$

- Free Differential Algebras a.k.a. (dual of) *Lie$_n$ algebras* (Sullivan, Castellani-d'Auria-Fré): given $\omega \in H^n(\mathfrak{g})$, we introduce a new generator η^{n-1} s.t.

$$d\eta^{i(n-1)} = \omega^{i(n)} \ .$$

- Sullivan's theorems: - Every free differential algebra is isomorphic to the tensor product of a unique minimal algebra and a unique contractible algebra.
  -a minimal differential algebra is determined by the dual Lie algebra defined by the quadratic equations and its cohomology classes that determine the differential algebra extension.
- L_∞-algebras: given $\omega^{i(n)} = C^i_{i_1 \ldots i_n} V^{i_1} \wedge \ldots \wedge V^{i_n}$, then

$$d\eta^{i(n-1)} = \omega^{i(n)} \iff [X_{i_1}, \ldots, X_{i_n}] = C^i_{i_1 \ldots i_n} \tilde{\eta}_i \ .$$

  Addition of *n-ary* product; L_∞ conditions automatically satisfied as $d^2 = 0$.
- Higher algebras $\iff$ Supergravity multiplets
  EX: flat $d = 6, N = (2,0)$, whose field content is

$$\left(V^a, \psi^A_\alpha, \omega^{ab}, B^{AB} \right)$$

  B comes from the $3-$cocycle.

- Branes and higher WZW models (Fiorenza, Sati, Schreiber):

$$d\eta^{(n-1|0)} = \omega^{(n|0)} \ , \ \exp\left(i \int_{\Sigma^{n-1}} \mathcal{L}_{WZW} \right) = \exp\left(i \int_{\Sigma^{n-1}} \eta^{(n-1)} \right) \ ;$$

iterate the FDA construction to identify new generators.

- Goal: extend to unexplored form complexes in the graded setting: pseudoforms.

$$d\eta^{(n-1|q)} = \omega^{(n|q)} \quad \overset{?}{\Longrightarrow} \quad \Sigma^{(p-1|q)} \ .$$

Superbranes? New supergravity couplings? Relation to double superstring/superbrane (see Sorokin, *superembedding*)?

$$\begin{aligned} \phi \quad &: \quad \Sigma^{(p|q)} \longrightarrow \mathcal{M}^{(m|n)} \ , \\ S_{double} \quad &= \quad \int_{\Sigma^{(p|q)}} (V^\mu)^* \wedge \star (V^\nu)^* \, g_{\mu\nu} \ , \\ \star \quad &: \quad \Omega^{(a|b)}\left(\Sigma^{(p|q)}\right) \longrightarrow \Omega^{(p-a|q-b)}\left(\Sigma^{(p|q)}\right) \ . \end{aligned}$$

A *super ringed space* (*superspace*) X is a topological space $|X|$ endowed with a sheaf of supercommutative rings $\mathcal{O}_X$:

$$X = (|X|, \mathcal{O}_X) \ .$$

Example

We can take $|X| = \mathbb{R}^p$, $\mathcal{O}_X = C^\infty_{\mathbb{R}^p}\left[\theta^1, \ldots, \theta^q\right]$ so that

$$\mathbb{R}^{(p|q)} := \left(\mathbb{R}^p, C^\infty_{\mathbb{R}^p}\left[\theta^1, \ldots, \theta^q\right]\right) \ .$$

A *(smooth) supermanifold* $\mathcal{SM} = (|\mathcal{SM}|, \mathcal{O}_{\mathcal{SM}})$ is a superspace which is locally isomorphic to $\mathbb{R}^{p|q}$: $\forall x \in |\mathcal{SM}|$, $\exists V \subseteq |\mathcal{SM}|$ such that $x \in V$ and

$$\mathcal{O}_{\mathcal{SM}}|_V \cong C^\infty_{\mathbb{R}^p}|_V (t^1, \ldots, t^p) \left[\theta^1, \ldots, \theta^q\right] \ .$$

We call $\left\{t^1, \ldots, t^p; \theta^1, \ldots, \theta^q\right\}$ the *supercoordinates of* $\mathcal{SM}$ *in* V and denote by $\dim \mathcal{SM} = (p|q)$ the *superdimension* of $\mathcal{SM}$.

- Given $\mathcal{SM} = (|\mathcal{SM}|, \mathcal{O}_{\mathcal{SM}})$, we call *nilpotent sheaf* $\mathcal{J}_{\mathcal{SM}}$ the sheaf of ideals of $\mathcal{O}_{\mathcal{SM}}$ generated by all the nilpotent sections, i.e., $\mathcal{J}_{\mathcal{SM}} = \mathcal{O}_{\mathcal{SM},1} \oplus \mathcal{O}^2_{\mathcal{SM},1}$.
- We call *reduced manifold* the manifold $\mathcal{SM}_{red} = (|\mathcal{SM}|, \mathcal{O}_{\mathcal{SM}}/\mathcal{J}_{\mathcal{SM}})$. Loosely speaking, "set the odd sections to zero".
- It always exists a canonical inclusion $i : \mathcal{SM}_{red} \hookrightarrow \mathcal{SM}$, which can be realised locally as

$$(x_1, \dots, x_m) \quad \mapsto \quad (x_1, \dots, x_m, 0, \dots, 0) \ .$$

- On the structure sheaf side, we have

$$0 \longrightarrow \mathcal{J}_{\mathcal{SM}} \longrightarrow \mathcal{O}_{\mathcal{SM}} \underset{p^*(?)}{\overset{i^*}{\rightleftarrows}} \mathcal{O}_{\mathcal{SM}_{red}} \longrightarrow 0 \ .$$

If such a projection exists, it is non-canonical and the supermanifold is called *projected*. This short exact sequence extends to superforms as

$$0 \longrightarrow Ker^{(p|0)}(i^*) \longrightarrow \Omega^{(p|0)}(\mathcal{SM}) \underset{p^*}{\overset{i^*}{\rightleftarrows}} \Omega^{(p)}(\mathcal{SM}_{red}) \longrightarrow 0 \ .$$

This leads to the factorisation $\Omega^{(p|0)}(\mathcal{SM}) = \Omega^{(p)}(\mathcal{SM}_b) \oplus Ker^{(p|0)}(i^*)$, thus splitting forms in "body" and "soul".

Forms on Supermanifolds

Let $\mathcal{SM} = \left(|\mathcal{SM}|, C^\infty_{\mathbb{R}^m}\left[\theta^1, \ldots, \theta^n\right] \right)$ of dimension $dim\mathcal{SM} = (m|n)$ be a (smooth) supermanifold.

- complex of *superforms*: $\left(\Omega^{(\bullet|0)} (\mathcal{SM}, d_{dR}) \right)$. It is *unbounded from above*:

$$0 \xrightarrow{\mathrm{d}} \Omega^{(0|0)}_{\mathcal{SM}} \xrightarrow{\mathrm{d}_{dR}} \Omega^{(1|0)}_{\mathcal{SM}} \xrightarrow{\mathrm{d}_{dR}} \ldots \xrightarrow{\mathrm{d}_{dR}} \Omega^{(m|0)}_{\mathcal{SM}} \xrightarrow{\mathrm{d}_{dR}} \ldots$$

as a consequence of the commutation relations

$$dx^i \wedge dx^j = -dx^j \wedge dx^i \ , \ dx^i \wedge d\theta^\alpha = d\theta^\alpha \wedge dx^i \ , \ d\theta^\alpha \wedge d\theta^\beta = d\theta^\beta \wedge d\theta^\alpha \ .$$

The notion of *top form* has to be found in the *Berezinian line bundle*, the super-analogous of the *Determinant line bundle*. With the Berezinian bundle one defines the complex of *integral forms*, which is unbounded from below:

$$\ldots \xrightarrow{\delta} \Omega^{(0|n)}_{\mathcal{SM}} \xrightarrow{\delta} \ldots \xrightarrow{\delta} \Omega^{(m-1|n)}_{\mathcal{SM}} \xrightarrow{\delta} \Omega^{(m|n)}_{\mathcal{SM}} \xrightarrow{\delta} 0 \ ,$$

where we denoted $\Omega^{(p|n)}_{\mathcal{SM}} := \mathcal{B}er(\mathcal{SM}) \otimes S^{n-p}(\Pi\mathcal{T}(\mathcal{SM}))$.

Berezinian: heuristic introduction

Consider the standard setting, one can construct a Clifford algebra out of forms and contractions (recall $|dx^i| = 1 = |\iota_i|$):

$$\left\{ dx^i, dx^j \right\} = 0 = \{\iota_i, \iota_j\} \ , \ \left\{ \iota_i, dx^j \right\} = \delta^i_j \ .$$

We can construct irreducible modules which are all isomorphic:

$$\text{choose } \ |\downarrow\rangle \ \text{ s.t. } \ \iota_i \, |\downarrow\rangle = 0 \, \forall i \ ,$$

hence we can identify $|\downarrow\rangle = 1$ and all the other states (forms) are obtained as $dx^{i_1} \dots dx^{i_p} \, |\downarrow\rangle$.
Equivalently,

$$\text{choose } \ |\uparrow\rangle \ \text{ s.t. } \ dx^i \, |\uparrow\rangle = 0 \, \forall i \ ,$$

hence we can identify $|\uparrow\rangle = Det = \bigwedge_i dx^i$ and all the other states (forms) are obtained as $\iota_{i_1} \dots \iota_{i_p} \, |\downarrow\rangle$.
The two constructions are equivalent as $|\uparrow\rangle = dx^1 \dots dx^n \, |\downarrow\rangle$ and viceversa.

Consider now a super setting, $\mathbb{R}^{(0|n)}$. The $(1|0)$-forms $d\theta^\alpha$ are even, them from $d\theta^\alpha$ and the contractions ι_α we can construct a Weyl algebra

$$\left[d\theta^\alpha , d\theta^\beta \right] = 0 = \left[\iota_\alpha , \iota_\beta \right] \ , \ \left[\iota_\alpha , d\theta^\beta \right] = \delta_\alpha^\beta \ .$$

Now there are many irreducible modules:

$$\text{choose } \left| \downarrow \right\rangle \text{ s.t. } \iota_\alpha \left| \downarrow \right\rangle = 0 \, \forall \alpha \ ,$$

hence we can identify $\left| \downarrow \right\rangle = 1$ and all the other states (superforms) of this representation are obtained as $(d\theta^{\alpha_1})^{p_1} \ldots (d\theta^{\alpha_n})^{p_n} \left| \downarrow \right\rangle$. These forms will be called *superforms*.

A different module can be constructed by starting from

$$\text{choose } \left| \uparrow \right\rangle \text{ s.t. } d\theta^\alpha \left| \uparrow \right\rangle = 0 \, \forall \alpha \ ,$$

then we can identify $\left| \uparrow \right\rangle = Ber = \prod_\alpha \delta \left(d\theta^\alpha \right)$ and the other states are constructed by acting with contractions (which are formal derivatives for the δ's). These modules are inequivalent. These forms will be called *integral forms*.

One can also construct intermediate modules: consider $\mathbb{R}^{(0|2)}$, one could construct an intermediate state $|\uparrow\downarrow\rangle$ as

$$d\theta^1 |\uparrow\downarrow\rangle = 0 = \iota_2 |\uparrow\downarrow\rangle \ , \ |\uparrow\downarrow\rangle = \delta\left(d\theta^1\right) \ .$$

In this case, the module in "infinitely generated in both directions":

$$\left(d\theta^2\right)^p (\iota_1)^q \, \delta\left(d\theta^1\right) \ , \ \forall p, q \ .$$

- In this realisation, the number of δs is called *picture number*;
- these forms will be called *pseudoforms*;
- pseudoform complexes are unbounded both from above and from below.

Berezinian: Distributional Realisation

One way to realise integral forms is as (compactly supported) *generalised functions* on $\mathrm{Tot}\,\Pi\mathcal{T}^*(\mathcal{SM})$, that is elements

$$\omega(x^1, \dots, x^n, d\theta^1, \dots, d\theta^m | \theta^1, \dots, \theta^m, dx^1, \dots dx^n) \in \Pi\mathcal{T}(\mathscr{M}) \ ,$$

where $x^i|\theta^\alpha$ are local coordinates for $\mathcal{SM}$, which only allow a *distributional dependence* supported in $d\theta^1 = \ldots = d\theta^m = 0$. $(\bullet|n)$-integral forms are defined as

$$\mathscr{B}er\left(\mathcal{SM}\right) \otimes S^{n-p}(\Pi\mathcal{T}(\mathfrak{g})) \coloneqq \mathcal{O}_{\mathcal{SM}} \cdot \left(\iota_{\mathcal{Y}}\right)^{n-p} \cdot \left[\bigwedge_{i=1}^{\dim \mathcal{SM}_0} dx^i \wedge \bigwedge_{\alpha=1}^{\dim \mathcal{SM}_1} \delta\left(d\theta^\alpha\right) \right]$$

and it locally reads

$$\omega^{(p|n)} = \omega_{[i_1 \dots i_q j_1 \dots j_n]}\left(x, \theta\right) dx^{i_1} \wedge \dots dx^{i_q} \wedge \delta^{(j_1)}\left(d\theta^1\right) \wedge \ldots \wedge \delta^{(j_n)}\left(d\theta^n\right) \ .$$

We are denoting $\delta^{(j)}\left(d\theta^\alpha\right) \coloneqq \left(\iota_\alpha\right)^j \delta\left(d\theta^\alpha\right)$. These distribution satisfy the relations

$$\left|\delta^{(j)}\left(d\theta^\alpha\right)\right| = 1 \ , \ \forall j \in \mathbb{N} \cup \{0\} \ , \ \delta\left(d\theta^\alpha\right) \wedge \delta\left(d\theta^\beta\right) = -\delta\left(d\theta^\beta\right) \wedge \delta\left(d\theta^\alpha\right) \ ,$$

$$d\theta^\alpha \delta^{(j)}\left(d\theta^\alpha\right) = -j\delta^{(j-1)}\left(d\theta^\alpha\right) \ , \ \delta\left(\lambda d\theta^\alpha\right) = \frac{1}{\lambda}\delta\left(d\theta^\alpha\right) \ .$$

Berezinian: Polyvector Fields Realisation

The Berezinian is defined as

$$\mathcal{B}er\left(\mathcal{SM}\right) := \left(\mathcal{B}er\Omega^{1}_{odd}\left(\mathcal{SM}\right)\right)^{*} \ .$$

Let $x^i|\theta^\alpha$, $i = 1, \dots, n$ and $\alpha = 1, \dots, m$ be local coordinates, then

$$\mathcal{B}er\left(\mathcal{SM}\right) \cong \mathcal{O} \cdot \left[\bigwedge_{i=1}^{\dim \mathcal{SM}_{\mathbf{0}}} dx^i \otimes \bigwedge_{\alpha=1}^{\dim \mathcal{SM}_{\mathbf{1}}} \partial_{\theta^\alpha} \right] \ .$$

Integral forms are defined as $\Omega^{(p|n)}_{\mathcal{SM}} := \mathcal{B}er(\mathcal{SM}) \otimes S^{n-p}(\Pi\mathcal{T}(\mathcal{SM}))$ and they form a complex equipped with the differential δ defined via Lie derivative:

$$\delta = \sum_a \mathcal{L}^R_{\frac{\partial}{\partial z^a}} \otimes_{\mathbb{K}} \frac{\partial}{\partial\left(\pi \frac{\partial}{\partial z^a}\right)} (-1)^{|z^a|} \ .$$

Again, the complex is unbounded from below

$$\dots \xrightarrow{\delta} \Omega^{(0|n)}_{\mathcal{SM}} \xrightarrow{\delta} \dots \xrightarrow{\delta} \Omega^{(m-1|n)}_{\mathcal{SM}} \xrightarrow{\delta} \Omega^{(m|n)}_{\mathcal{SM}} \xrightarrow{\delta} 0 \ .$$

Distributional Realisation: Pseudoforms

The distributional realisation suggests the construction of a different type of forms, with *non-maximal* and *non-zero* number of delta's:

$$\omega^{(p|s)} = \omega_{[i_1 \ldots i_q j_1 \ldots j_n]} (x, \theta) \, dx^{i_1} \wedge \ldots dx^{i_q} \wedge \delta^{(j_1)} \left(d\theta^1 \right) \wedge \ldots \wedge \delta^{(j_s)} \left(d\theta^s \right) \ , \ 0 < s < n \ .$$

These objects are not well defined; for example, they do not behave tensorially

$$d\theta^\alpha \mapsto \Lambda^\alpha_\mu dx^\mu + \Lambda^\alpha_\beta d\theta^\beta \ , \ \delta \left(d\theta^\alpha \right) \mapsto \delta \left(\Lambda^\alpha_\mu dx^\mu + \Lambda^\alpha_\beta d\theta^\beta \right) = \ldots ?$$

Example

$$d\theta^1 \mapsto d\theta^1 + d\theta^2 \implies \delta \left(d\theta^1 \right) \mapsto \delta \left(d\theta^1 + d\theta^2 \right) = \sum_{i=0}^{\infty} \frac{\left(d\theta^2 \right)^i}{i!} \delta^{(i)} \left(d\theta^1 \right) \ .$$

Some hints to define pseudoforms: Manin, Witten.

Introducing Lie Algebra Cohomology

Let $\mathfrak{g}$ be an finite dimensional Lie (super)algebra defined over the field $\mathbb{K} = \mathbb{C}, \mathbb{R}$, and let V be a $\mathfrak{g}$-module; a *p-chain* of $\mathfrak{g}$ valued in V is an (graded)alternating $\mathbb{K}$-linear map

$$C_p(\mathfrak{g}, V) \coloneqq \wedge^p \mathfrak{g} \otimes V \ ,$$

where $\wedge^p \mathfrak{g}$ is for $\mathfrak{g}$ considered a vector space. This can be lifted to a complex by introducing the differential $\partial : C_p\left(\mathfrak{g}, V\right) \to C_{p-1}\left(\mathfrak{g}, V\right)$

$$\partial \left[f \otimes \left(\mathcal{Y}_{a_1} \wedge \ldots \wedge \mathcal{Y}_{a_p} \right) \right] = \sum_{i=1}^{p} (-1)^{\delta_i} \left(\mathcal{Y}_{a_i} f \right) \otimes \left(\mathcal{Y}_{a_1} \wedge \ldots \wedge \hat{\mathcal{Y}}_{a_i} \wedge \ldots \wedge \mathcal{Y}_{a_p} \right) + \\ + \sum_{i<j}^{p} (-1)^{\delta_{i,j}} f \otimes \left(\left[\mathcal{Y}_{a_i}, \mathcal{Y}_{a_j} \right] \wedge \mathcal{Y}_{a_1} \wedge \ldots \wedge \hat{\mathcal{Y}}_{a_i} \wedge \hat{\mathcal{Y}}_{a_j} \wedge \ldots \wedge \mathcal{Y}_{a_p} \right) \ .$$

On the dual side, we can define *p-cochains* as

$$C^p(\mathfrak{g}, V) := Hom_{\mathbb{K}}(\wedge^p \mathfrak{g}, V) = \wedge^p \mathfrak{g}^* \otimes V = \bigoplus_{r=1}^{p} \left(\wedge^r \mathfrak{g}_0^* \otimes S^{q-r} \mathfrak{g}_1^*\right) \otimes V \ .$$

Again, we can introduce a differential $d : C^p(\mathfrak{g}, V) \to C^{p+1}(\mathfrak{g}, V)$

$$d\omega\left(\mathcal{Y}_{a_1}, \ldots, \mathcal{Y}_{a_{p+1}}\right) = \sum_{i=1}^{p+1} (-1)^{\delta_i} \mathcal{Y}_{a_i} \omega\left(\mathcal{Y}_{a_1} \wedge \ldots \wedge \hat{\mathcal{Y}}_{a_i} \wedge \ldots \wedge \mathcal{Y}_{a_{p+1}}\right) + \\ + \sum_{i<j} (-1)^{\delta_{i,j}} \omega\left(\left[\mathcal{Y}_{a_i}, \mathcal{Y}_{a_j}\right] \wedge \mathcal{Y}_{a_1} \wedge \ldots \wedge \hat{\mathcal{Y}}_{a_i} \wedge \ldots \wedge \hat{\mathcal{Y}}_{a_j} \wedge \ldots \wedge \mathcal{Y}_{a_{p+1}}\right) \ .$$

$$d^2 = 0 \iff (graded) Jacobi$$

Relative Lie (super)algebra cohomology

Given a Lie (super)algebra $\mathfrak{g}$ and a Lie sub-(super)algebra $\mathfrak{h}$ (we denote $\mathfrak{k} = \mathfrak{g}/\mathfrak{h}$), we define the space of *horizontal p-cochains* with values in a module V as

$$C^p(\mathfrak{k}, V) := \{\omega \in C^p(\mathfrak{g}, V) : \iota_{\mathcal{Y}_a}\omega = 0, \forall \mathcal{Y}_a \in \mathfrak{h}\} \ ,$$

We define the space of $\mathfrak{h}$*-invariant p-cochains* with values in V as

$$(C^p(\mathfrak{g}, V))^{\mathfrak{h}} := \{\omega \in C^p(\mathfrak{g}, V) : \mathcal{L}_{\mathcal{Y}_a}\omega = 0, \forall \mathcal{Y}_a \in \mathfrak{h}\} \ .$$

Forms which are both horizontal and $\mathfrak{h}$-invariant are called *basic*:

$$(C^p(\mathfrak{k}, V))^{\mathfrak{h}} \equiv (C^p(\mathfrak{g}/\mathfrak{h}, V))^{\mathfrak{h}} \ .$$

We define the *cohomology of* $\mathfrak{g}$ *relative to* $\mathfrak{h}$ as

$$H^{\bullet}(\mathfrak{g}, \mathfrak{h}, V) := \frac{\left\{\omega \in (C^{\bullet}(\mathfrak{k}, V))^{\mathfrak{h}} : \nabla_{\mathfrak{k}}\omega = 0\right\}}{\left\{\omega \in (C^{\bullet}(\mathfrak{k}, V))^{\mathfrak{h}} : \exists \eta \in (C^{\bullet}(\mathfrak{k}, V))^{\mathfrak{h}}, \omega = \nabla_{\mathfrak{k}}\eta\right\}} \ ,$$

where the differential is

$$d\big|_{basic} \equiv \nabla_{\mathfrak{k}} \ .$$

Haar Berezinian

Let $\mathcal{Y}_A = \{X_i, \chi_\alpha\}$ and $\mathcal{Y}^{*A} = \left\{V^i, \psi^\alpha\right\}$ the bases of vectors and MC forms, respectively, the *Haar Berezinian* reads

$$\mathcal{B}er\left(\mathfrak{g}\right) := V \cdot \left[\bigwedge_{i=1}^{m} V^i \otimes \bigwedge_{\alpha=1}^{n} \xi_\alpha\right] \equiv V \cdot \mathcal{D} \ .$$

Integral forms are then defined as

$$C^p_{int}(\mathfrak{g}) := \mathcal{B}er(\mathfrak{g}) \otimes S^{n-p}(\Pi\mathcal{T}(\mathfrak{g})) \ ,$$
$$\delta\left(\mathcal{D} \otimes \left[f \otimes \mathcal{Y}_1 \wedge \ldots \wedge \mathcal{Y}_{m-p}\right]\right) = \mathcal{D} \otimes \partial\left[f \otimes \mathcal{Y}_1 \wedge \ldots \wedge \mathcal{Y}_{m-p}\right] \ .$$

In the distributional realisation, the differential δ is defined as $\delta = d$, acting on integral forms via the previous formal relations. The *Haar Berezinian* is now defined as

$$\mathcal{B}er\left(\mathfrak{g}\right) := V \cdot \left[\bigwedge_{i=1}^{m} V^i \otimes \bigwedge_{\alpha=1}^{n} \delta\left(\psi^\alpha\right)\right] \equiv V \cdot \mathcal{D} \equiv \omega^{top}_{\mathfrak{g}} \ .$$

Any integral form can be obtained by acting on $\omega^{top}_{\mathfrak{g}}$ with contractions:

$$\omega^{(m-p|n)} = \iota_{\mathcal{Y}_{A_1}} \ldots \iota_{\mathcal{Y}_{A_p}} \omega^{top}_{\mathfrak{g}} \ ,$$

thus reproducing the structure $\mathcal{B}er(\mathcal{SM}) \otimes S^{n-p}(\Pi\mathcal{T}(\mathcal{SM}))$.

Dictionary: Dirac $\leftrightsquigarrow$ Koszul

In both realisations we have that integral forms are defined as

$$C^p_{int}(\mathfrak{g}, V) := \mathcal{B}er(\mathfrak{g}, V) \otimes S^p \Pi \mathfrak{g} \ .$$

We have

$$\begin{aligned} \mathcal{B}er : V \cdot \left(\bigwedge_{i=1}^m \mathcal{V}^i \right) \left(\bigwedge_{\alpha=1}^m \delta(\psi^\alpha) \right) \quad &\leftrightsquigarrow \quad V \cdot \left[\bigwedge_{i=1}^m \mathcal{V}^i \otimes \bigwedge_{\alpha=1}^m \chi_\alpha \right] \\ \omega^{m-p}_{int} \left(\equiv \omega^{(m-p|n)} \right) : \left(\prod_{i=1}^p \iota_{\mathcal{Y}_i} \right) \mathcal{B}er \quad &\leftrightsquigarrow \quad \mathcal{B}er \otimes \left(\bigwedge_{i=1}^p \pi \mathcal{Y}^i \right) \\ d_{CE} : d \quad &\leftrightsquigarrow \quad \delta = 1 \otimes \partial \end{aligned}$$

Pseudoforms as Infinite-Dimensional Representations

Given $\mathfrak{g}$, $\dim \mathfrak{g} = (m|n)$, $\mathfrak{h}$, $\dim \mathfrak{h} = (p|q)$ and $\mathfrak{k} = \mathfrak{g}/\mathfrak{h}$, $\dim \mathfrak{k} = (m-p|n-q)$, we can define

$$\mathcal{B}er\left(\mathfrak{h}\right) := \mathbb{K} \cdot \left[\bigwedge_{i=1}^{p} V^i \wedge \bigwedge_{\alpha=1}^{q} \delta\left(\psi^\alpha\right) \right] \ , \ V^i, \psi^\alpha \in \Pi\mathfrak{h}^* \ ,$$

$$\mathcal{B}er\left(\mathfrak{k}\right) := \mathbb{K} \cdot \left[\bigwedge_{\hat{i}=1}^{m-p} V^{\hat{i}} \wedge \bigwedge_{\hat{\alpha}=1}^{n-q} \delta\left(\psi^{\hat{\alpha}}\right) \right] \ , \ V^{\hat{i}}, \psi^{\hat{\alpha}} \in \Pi\mathfrak{k}^* \ .$$

They are not $\mathfrak{g}$-modules. We can use them to construct $\mathfrak{g}$-modules:

$$V_{\mathfrak{h}}^{(p|q)} := \bigoplus_{i=0}^{\infty} \left(S^i \Pi\mathfrak{h} \otimes \mathcal{B}er\left(\mathfrak{h}\right) \right) \otimes S^i \Pi\mathfrak{k}^* = \bigoplus_{i=0}^{\infty} C_{int}^{m-i}\left(\mathfrak{h}\right) \otimes C^i\left(\mathfrak{k}\right) \ ,$$

$$V_{\mathfrak{k}}^{(m-p|n-q)} := \bigoplus_{i=0}^{\infty} \left(S^i \Pi\mathfrak{k} \otimes \mathcal{B}er\left(\mathfrak{k}\right) \right) \otimes S^i \Pi\mathfrak{h}^* = \bigoplus_{i=0}^{\infty} C_{int}^{m-p-i}\left(\mathfrak{k}\right) \otimes C^i\left(\mathfrak{h}\right) \ .$$

Then pseudoforms are

$$C^{p\pm s}\left(\mathfrak{g}, V_{\mathfrak{h}}^{(p|q)}\right) \equiv C^{(p\pm s|q)}(\mathfrak{g})\ ,\ C^{m-p\pm s}\left(\mathfrak{g}, V_{\mathfrak{k}}^{(m-p|n-q)}\right) \equiv C^{(m-p\pm s|n-q)}(\mathfrak{g})\ .$$

They can be naturally equipped with a differential constructed from d and ∂. In the distributional realisation we have (consider $V = \mathbb{K}$)

$$\begin{aligned} d : C^{(s|\bullet)}(\mathfrak{g}) &\rightarrow C^{(s+1|\bullet)}(\mathfrak{g}) \\ \omega &\mapsto d\omega = \sum_{a,b,c} f^{a}_{bc}\mathcal{Y}^{*b} \wedge \mathcal{Y}^{*c} \wedge \iota_{\mathcal{Y}_c}\omega\ . \end{aligned}$$

The pseudoform cohomology is defined as

$$\begin{aligned} H^{(\bullet|q)}(\mathfrak{g}) &:= H^{\bullet}\left(\mathfrak{g}, V_{\mathfrak{h}}^{(p|q)}\right)\ , \\ H^{(\bullet|n-q)}(\mathfrak{g}) &:= H^{\bullet}\left(\mathfrak{g}, V_{\mathfrak{k}}^{(m-p|n-q)}\right)\ . \end{aligned}$$

Theorem

If $m \geq n$, the natural inclusions

$$\begin{aligned} \mathfrak{gl}(m) &\to \mathfrak{gl}(m|n)_0 &\subset \mathfrak{gl}(m|n) \ , \\ \mathfrak{sl}(m) &\to \mathfrak{sl}(m|n)_0 &\subset \mathfrak{sl}(m|n) \ , \end{aligned}$$

induce an isomorphism in cohomology with trivial coefficients.

Theorem

$$H^\bullet(\mathfrak{osp}(m|n)) = \begin{cases} H^\bullet(\mathfrak{so}(m)) \ , & \text{if } m \geq 2n \ , \\ H^\bullet(\mathfrak{sp}(n)) \ , & \text{if } m < 2n \ . \end{cases}$$

RMK

Only a part of the bosonic subalgebra contributes to the CE cohomology. The CE cohomology is related to the superalgebra invariants (or analogously to its rank): it looks like "some invariants get lost".

Cartan's Theorem

Under the topological assumptions of *compactness* and *connectedness*, the de Rham cohomology of a Lie group G is isomorphic to the cohomology of its Lie algebra (valued in the real numbers), i.e., $H^p_{dR}(G) \cong H^p_{CE}(\mathfrak{g})$. This fails in the supersetting: one has

$$H^p_{dR}\left(U(1|1)\right) = H^p_{dR}\left(U(1|1)_{red}\right) =$$

$$= H^p_{dR}\left(U(1) \times U(1)\right) = \begin{cases} \mathbb{R} \ , \ \text{if } p = 0, 2 \ , \\ \Pi\mathbb{R}^2 \ , \ \text{if } p = 1 \ , \\ 0 \ , \ \text{else} \ . \end{cases}$$

This shows that the de Rham cohomology of *compact* Lie supergroups, such as for example $U(1|1)$ which is topologically a 2-torus, is not isomorphic to the Chevalley-Eilenberg cohomology of superforms of their related Lie superalgebras.

The Berezinian Complement Isomorphism

We define the "Berezinian complement" map $\star$ as

$$\begin{aligned} \star : C^p(\mathfrak{g}) &\to C^{m-p}_{int}(\mathfrak{g}) \\ \omega &\mapsto \star\omega^{(p|0)} = (\star\omega)^{(m-p|n)} := \left(\prod_{i=1}^{p} \iota_{\mathcal{Y}_{A_i}}\right) \omega^{top}_{\mathfrak{g}} \ , \end{aligned}$$

where $\left(\prod_{i=1}^{p} \iota_{\mathcal{Y}_{A_i}}\right) \omega = 1$. This map induces a cohomology isomorphism:

$$\star : H^\bullet(\mathfrak{g}) \xrightarrow{\cong} H^{m-\bullet}_{int}(\mathfrak{g}) \ .$$

The isomorphism is verified when $\mathfrak{g}$ admits non-degenerate invariant bilinear form, hence it holds e.g. for "basic Lie superalgebras".

Spectral Sequences

The idea is to reconstruct the cohomology of a Lie algebra starting from the (eventually known) cohomology of substructure.

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k} \ , \ \mathfrak{k} = \mathfrak{g}/\mathfrak{h} \ , \ \mathfrak{h} \text{ sub-algebra.}$$

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h} \ , \ [\mathfrak{h}, \mathfrak{k}] \subseteq \mathfrak{h} + \mathfrak{k} \ , \ [\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{h} + \mathfrak{k} \ .$$

The cohomology is calculated via *approximations*: we can split the differential d as

$$d = d_0 + d_1 + d_2 + \dots$$

then, calculate the cohomology w.r.t. d_0, then d_1, then d_2 etc., up to *convergence*.

- Trivial modules: Koszul
- General modules: Hochschild-Serre

Given a Lie algebra $\mathfrak{g}$ and a Lie sub-algebra $\mathfrak{h}$, we define the filtration

$$F^p C^q(\mathfrak{g}) = \left\{ \omega \in C^q(\mathfrak{g}) : \forall \xi_i \in \mathfrak{h}, \iota_{\xi_{i_1}} \dots \iota_{\xi_{i_{q+1-p}}} \omega = 0 \right\} ,$$

which is a filtration in the sense that

$$dF^p C^q(\mathfrak{g}) \subseteq F^p C^{q+1}(\mathfrak{g}) \ , \ \forall p, q \in \mathbb{Z} \ .$$

There exists a spectral sequence $(E_s^{\bullet,\bullet}, d_s), d_s : E_s^{p,q} \to E_s^{p+s,q+1-s}$ that converges to $H(\mathfrak{g})$. The first space (page zero) reads

$$E_0^{m,n} := F^m C^{(m+n)}(\mathfrak{g}) / F^{m+1} C^{(m+n)}(\mathfrak{g}) \ .$$

The differentials d_s are induced by the CE differential:

$$d = V_{\mathfrak{h}} V_{\mathfrak{h}} \iota_{\mathfrak{h}} + V_{\mathfrak{h}} V_{\mathfrak{k}} \iota_{\mathfrak{h}} + V_{\mathfrak{k}} V_{\mathfrak{k}} \iota_{\mathfrak{h}} + V_{\mathfrak{h}} V_{\mathfrak{k}} \iota_{\mathfrak{k}} + V_{\mathfrak{k}} V_{\mathfrak{k}} \iota_{\mathfrak{k}} \ ,$$

reflecting the structure

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \ , \ [\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{h} + \mathfrak{k} \ , \ [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{h} + \mathfrak{k} \ .$$

The following pages of the spectral sequence are defined as

$$E_s^{\bullet,\bullet} := H\left(E_{s-1}^{\bullet,\bullet}, d_{s-1}\right) \ .$$

In particular, the first differential formally reads

$$d_0 = V_{\mathfrak{h}} V_{\mathfrak{h}} \iota_{\mathfrak{h}} + V_{\mathfrak{h}} V_{\mathfrak{k}} \iota_{\mathfrak{k}} \ .$$

Theorem (Koszul, Hochschild-Serre)

If $\mathfrak{h}$ is reductive in $\mathfrak{g}$ (i.e., $[\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}$), hence

$$E_1^{m,n} = H^n(\mathfrak{h}) \otimes \left(\Omega^m(\mathfrak{k})\right)^{\mathfrak{h}} \ .$$

$$E_2^{m,n} = H^n(\mathfrak{h}) \otimes H^m(\mathfrak{g}, \mathfrak{h}) \ .$$

Fuks: superform cohomology of classical Lie superalgebras with $\mathfrak{h} = \mathfrak{g}_0 \implies$ no pseudoforms. New objects can be found by repeating the procedure for *sub-superalgebras*. We can introduce two *inequivalent* filtrations

$$F^p C^{(q|l)}(\mathfrak{g}, V) := \left\{ \omega \in C^{(q|l)}(\mathfrak{g}, V) : \forall \mathcal{Y}_a \in \mathfrak{h}, \iota_{\mathcal{Y}_{a_{\mathbf{1}}}} \ldots \iota_{\mathcal{Y}_{a_{q+\mathbf{1}-p}}} \omega = 0 \right\} ,$$

$$\tilde{F}^p C^{(q|l)}(\mathfrak{g}, V) := \left\{ \omega \in C^{(q|l)}(\mathfrak{g}, V) : \forall \mathcal{Y}^{*a} \in \mathfrak{h}^*, \mathcal{Y}^{*a_{\mathbf{1}}} \wedge \ldots \wedge \mathcal{Y}^{*a_{q+\mathbf{1}-p}} \wedge \omega = 0 \right\} ,$$

with $p, q \in \mathbb{Z}$ and $l \in \{0, \dim \mathfrak{h}_1, \dim \mathfrak{k}_1, \dim \mathfrak{g}_1\}$.

- The two filtrations coincide if $\mathfrak{g}$ is a bosonic Lie algebra;
- if $\mathfrak{h}$ has non-trivial odd part, the second filtration is empty on superforms;
- If $\mathfrak{h}$ has non-trivial odd part, the first filtration is empty on integral forms, which are then kept into account by the second filtration only.

Page zero of the spectral sequence is defined, for any l, as

$$\mathcal{E}_0^{m,n} := E_0^{m,n} \oplus \tilde{E}_0^{m,n} := \frac{F^m \Omega^{(m+n|l)}(\mathfrak{g})}{F^{m+1}\Omega^{(m+n|l)}(\mathfrak{g})} \oplus \frac{\tilde{F}^{m+2n-r}\Omega^{(m+n|l)}(\mathfrak{g})}{\tilde{F}^{m+2n-r+1}\Omega^{(m+n|l)}(\mathfrak{g})} \ .$$

and on pseudoform complexes we obtain

$$l = \dim \mathfrak{h}_1 \quad \Longrightarrow \quad \mathcal{E}_0^{m,n} = \tilde{E}_0^{m,n} = C^{(m|0)}(\mathfrak{k}) \otimes C^{(n|\dim \mathfrak{h}_1)}(\mathfrak{h}) \ ,$$
$$l = \dim \mathfrak{k}_1 \quad \Longrightarrow \quad \mathcal{E}_0^{m,n} = E_0^{m,n} = C^{(m|\dim \mathfrak{k}_1)}(\mathfrak{k}) \otimes C^{(n|0)}(\mathfrak{h}) \ .$$

If $\dim \mathfrak{h}_1 = \dim \mathfrak{k}_1 = \dim \mathfrak{g}_1/2$, we have

$$\mathcal{E}_0^{m,n} = \left[C^{(m|\dim \mathfrak{g}_1/2)}(\mathfrak{k}) \otimes C^{(n|0)}(\mathfrak{h}) \right] \oplus \left[C^{(m|0)}(\mathfrak{k}) \otimes C^{(n|\dim \mathfrak{g}_1/2)}(\mathfrak{h}) \right] \ .$$

We construct a spectral sequence

$$d_s : \mathcal{E}_s^{p,q} \to \mathcal{E}_s^{p+s,q+1-s} \ , \ \mathcal{E}_{s+1}^{\bullet,\bullet} := \left(\mathcal{E}_s^{\bullet,\bullet}, d_s \right) \ , \ \mathcal{E}_\infty^{\bullet,\bullet} \cong H^{(\bullet|l)}(\mathfrak{g}, V) \ ,$$

and extend KHS theorem.

Proposition

Let $\mathfrak{g}$ *be a Lie (super)algebra over a (characteristic-zero) field* $\mathbb{K}$ *and let* $\mathfrak{h}$ *be a Lie sub-(super)algebra reductive in* $\mathfrak{g}$*,* $\dim \mathfrak{h}_1 \neq 0$*, and denote* $\mathfrak{k} = \mathfrak{g}/\mathfrak{h}$*. Then, at picture number* $l = \dim \mathfrak{h}_1, \dim \mathfrak{k}_1$*, if* $\dim \mathfrak{h}_1 \neq \dim \mathfrak{k}_1$*, the first pages of the extended spectral sequence read*

$$l = \dim \mathfrak{h}_1 \implies \mathcal{E}_1^{m,n} = \left(C^{(m|0)}(\mathfrak{k}) \right)^{\mathfrak{h}} \otimes H^{(n|\dim \mathfrak{h}_{\mathbf{1}})}(\mathfrak{h}) \ ,$$

$$\mathcal{E}_2^{m,n} = H^{(m|0)}(\mathfrak{g}, \mathfrak{h}) \otimes H^{(n|\dim \mathfrak{h}_{\mathbf{1}})}(\mathfrak{h}) \ ,$$

$$l = \dim \mathfrak{k}_1 \implies \mathcal{E}_1^{m,n} = \left(C^{(m|\dim \mathfrak{k}_{\mathbf{1}})}(\mathfrak{k}) \right)^{\mathfrak{h}} \otimes H^{(n|0)}(\mathfrak{h}) \ ,$$

$$\mathcal{E}_2^{m,n} = H^{(m|\dim \mathfrak{k}_{\mathbf{1}})}(\mathfrak{g}, \mathfrak{h}) \otimes H^{(n|0)}(\mathfrak{h}) \ .$$

If $\dim \mathfrak{h}_1 = \dim \mathfrak{k}_1 = \dim \mathfrak{g}_1/2$*, then the first two pages at picture number* $l = \dim \mathfrak{g}_1/2$ *read*

$$\mathcal{E}_1^{m,n} = \left[\left(C^{(m|0)}(\mathfrak{k}) \right)^{\mathfrak{h}} \otimes H^{(n|\dim \mathfrak{g}_{\mathbf{1}}/2)}(\mathfrak{h}) \right] \oplus \left[\left(C^{(m|\dim \mathfrak{g}_{\mathbf{1}}/2)}(\mathfrak{k}) \right)^{\mathfrak{h}} \otimes H^{(n|0)}(\mathfrak{h}) \right] \ ,$$

$$\mathcal{E}_2^{m,n} = \left[H^{(m|0)}(\mathfrak{g}, \mathfrak{h}) \otimes H^{(n|\dim \mathfrak{g}_{\mathbf{1}}/2)}(\mathfrak{h}) \right] \oplus \left[H^{(m|\dim \mathfrak{g}_{\mathbf{1}}/2)}(\mathfrak{g}, \mathfrak{h}) \otimes H^{(n|0)}(\mathfrak{h}) \right] \ .$$

Example ($\mathfrak{g} = \mathfrak{osp}(2|2)$)

4 even generators, 4 odd generators
From Fuks', we have

$$H^{(\bullet|0)}\left(\mathfrak{osp}(2|2)\right) = H^{\bullet}\left(\mathfrak{sp}(2)\right) = \left\{1, \omega^{(3)}\right\} \ .$$

The abelian factor $\mathfrak{so}(2)$ is the "lost part". We can choose

$$\mathfrak{h} = \mathfrak{osp}(1|2) \ , \ \text{3 bosons, 2 fermions} \quad \Longrightarrow \quad \text{picture 2 integral forms;}$$
$$\mathfrak{k} = \mathfrak{g}/\mathfrak{h} \ , \ \text{1 boson, 2 fermions} \quad \Longrightarrow \quad \text{picture 2 integral forms.}$$

One finds

$$H^{(\bullet|2)}\left(\mathfrak{osp}(2|2)\right) = \left\{\omega^{(0|2)}, \omega^{(1|2)}, \omega^{(3|2)}, \omega^{(4|2)}\right\} \ ,$$

$\omega^{(1|2)}$ encodes the abelian factor.

Example ($\mathfrak{g} = \mathfrak{osp}(1|4)$)

4 even generators, 4 odd generators
From Fuks', we have

$$H^{(\bullet|0)}\left(\mathfrak{osp}(1|4)\right) = H^{\bullet}\left(\mathfrak{sp}(4)\right) = \left\{1, \omega^{(3)}, \omega^{(7)}, \omega^{(10)}\right\} \ .$$

If we choose

$$\mathfrak{h} = \mathfrak{osp}(1|2) \times \mathfrak{sp}(2) \ , \ \text{6 bosons, 2 fermions} \implies \text{picture 2 integral forms;}$$
$$\mathfrak{k} = \mathfrak{g}/\mathfrak{h} \ , \ \text{4 boson, 2 fermions} \implies \text{picture 2 integral forms.}$$

One finds

$$H^{(\bullet|2)}\left(\mathfrak{osp}(2|2)\right) = \left\{\omega^{(0|2)\otimes 2}, \omega^{(3|2)\otimes 2}, \omega^{(7|2)\otimes 2}, \omega^{(10|2)\otimes 2}\right\} \ .$$

RMK: these classes *are not* $\mathfrak{osp}(1|4)$*-invariant.*

Theorem (Chevalley-Eilenberg)

If $\mathfrak{g}$ is a semi-simple Lie algebra and V a finite-dimensional module, then $H^p(\mathfrak{g}, V) = H^p(\mathfrak{g}, V^{\mathfrak{g}})$.

Theorem (Chevalley-Eilenberg)

If $\mathfrak{g}$ is a semi-simple Lie algebra with values over a characteristic zero field $\mathbb{K}$, every cohomology class $H^q(\mathfrak{g}, \mathbb{K})$ contains a $\mathfrak{g}$-invariant cocycle.

- Extension to superalgebras in the complexes of superforms and integral forms. Failure for pseudoforms: *infinite dimensional representations.*
- There are still some invariant cases, e.g., $\mathfrak{osp}(2|2)$ pseudoforms induced by $\mathfrak{osp}(1|2)$.
- Different choices of sub-superalgebras do not affect superforms and integral forms, but of course pseudoforms: different choices induce different pseudoforms.
- The choice of the sub-superalgebra corresponds to the choice of invariances of the cohomology classes.

Discussion

There are other ways to approach the problem:

- brute force; but pseudoforms live in infinite dimensional spaces, computations may be arbitrarily difficult
- Molien-Weyl integrals: it is possible to extend the bosonic formula to the super setting. Pseudoforms are not standard representations $\rightarrow$ infinite-dimensional representations $\rightarrow$ extremely rich, almost unexplored land

Outlook

- Extension of Sullivan's program to pseudoforms
- Classify cohomology groups among pseudoforms $\overset{?}{\Longrightarrow}$ new branes.
- Partially invariant pseudoforms $\overset{?}{\Longrightarrow}$ broken (super)symmetries/non-linear realisations of (super)symmetries?

Double superstring

$$
\begin{aligned}
\phi \quad &: \quad \Sigma^{(p|q)} \longrightarrow \mathcal{M}^{(m|n)} \ , \\
S_{double} \quad &= \quad \int_{\Sigma^{(p|q)}} (V^{\mu})^* \wedge \star (V^{\nu})^* \, g_{\mu\nu} + \int_{\Lambda^{p+\mathbf{1}|q}} \omega^{(p+\mathbf{1}|q)} \ , \\
\star \quad &: \quad \Omega^{(a|b)}\left(\Sigma^{(p|q)}\right) \longrightarrow \Omega^{(p-a|q-b)}\left(\Sigma^{(p|q)}\right) \ .
\end{aligned}
$$

What do they represent? Do they implement κ symmetry or other symmetries?