TALK DANG 29/09/2022

Joint work with Bailleul, Đạt , Ferdinand , Vignes-Tourneret.

Let (M,g) be a $\mathcal{C}^\infty$, closed, compact, Riemannian mfd.

Goal: construct the ϕ^4_3 measure on $\mathcal{D}'(M)$.

First non perturbative, interacting QFT on 3-mfd.

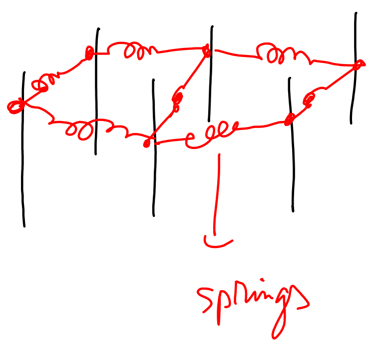
Method + Tools: SPDE technique, latest results of Gubinelli-Hofmanova
and Jagannath-Perkowski

+ The Epstein-Glaser technique developed by Popineau-Stora,
Brunetti-Fredenhagen
in the version of my Phd Thesis, μlocal analysis.

+ paradiff calculus of Bony-Meyer on mfds.

What is this all about?

<u>TOY MODEL</u>: discrete Box $\Lambda \subset \mathbb{Z}^3$



$$e^{-\overbrace{\sum_{i\sim j}\frac{|\sigma_i-\sigma_j|^2}{2}+\sum_{i\in\Lambda}\frac{m^2\sigma_i^2}{2}}^{S(\sigma)\ \text{Dirichlet action}}}\prod_{i\in\Lambda}d\sigma_i$$

Gibbs measure of massive

<u>DGFF</u>

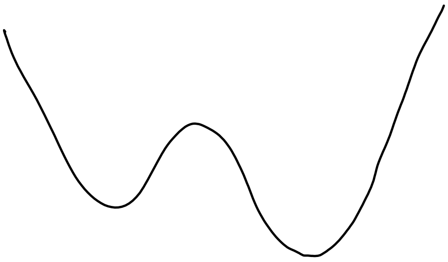
<u>Def</u>: [Correlation function]

$$\langle \sigma_{i_1}\dots\sigma_{i_k}\rangle_\Lambda = \frac{\int_{\mathbb{R}^\Lambda}\sigma_{i_1}\dots\sigma_{i_k}\,e^{-S(\sigma)}\,d^\Lambda\sigma}{\int_{\mathbb{R}^\Lambda}e^{-S(\sigma)}\,d^\Lambda\sigma}$$

$$= \frac{\int_{\mathbb{R}^\Lambda}\sigma_{i_1}\dots\sigma_{i_k}\,e^{\sum_{i\sim j}\boxed{\sigma_i\sigma_j}}\,d^\Lambda\sigma \quad \text{Ferromag interac}}{\int_{\mathbb{R}^\Lambda}e^{\sum_{i\sim j}\sigma_i\sigma_j}\,d^\Lambda\sigma}$$

Tendency of spins to have same signs!

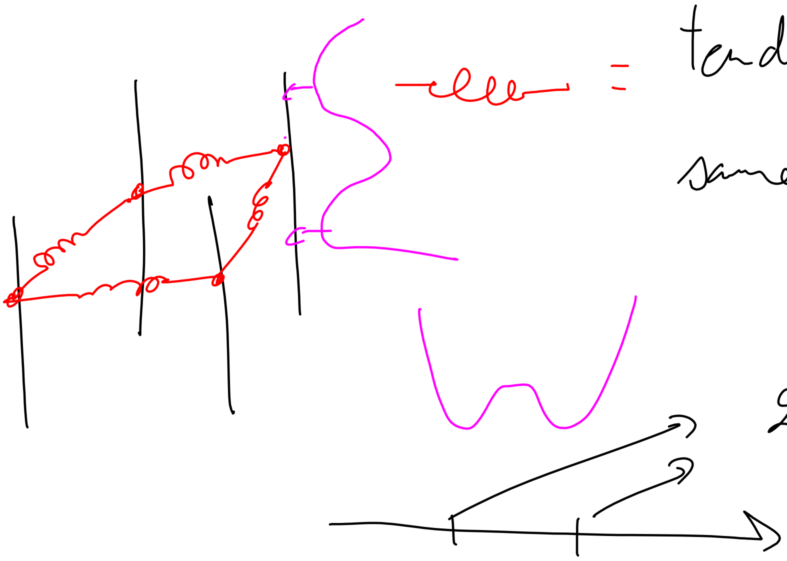
The $P(\sigma)$ discrete model,



P polynomial bounded from below

$$S(\sigma) = \sum_{i\sim j} \frac{|\sigma_i-\sigma_j|^2}{2} + \sum_i m^2\sigma_i^2 + \underbrace{P(\sigma_i)}_{\text{add polynomial}}$$

→ Intuitive meaning



= tendency to align masses same sign

2 preferred rest positions

$d=3$

Interested in $\langle \sigma_{i_1} \dots \sigma_{i_k} \rangle_\Lambda$ for $P(\sigma)=\sigma^4$

specially scaling limit

$$\langle \sigma_{n_1} \dots \sigma_{n_k} \rangle_{\varepsilon\Lambda} \xrightarrow[\varepsilon\to 0^+]{} \ ?$$

$$\frac{1}{\varepsilon^3}\sum_{i\in\varepsilon\Lambda}\sigma_i\,\delta_\varepsilon(i)\ \underset{\varepsilon\to 0^+}{\sim}$$ converges in law to random distribution?

Answer: Yes! ϕ^4_3 measure by Glimm-Jaffe 70's $\mathbb{R}^3, \mathbb{T}^3$

$P(\phi)_2$ Nelson, Segal 60's $\mathbb{R}^2, \mathbb{T}^2$

Many works: Fröhlich, Feldman, Rivasseau, Magnen, Sénéor, Brydges, Slade, Balaban, Gawedzki, Kupiainen, Spencer

In any case, ϕ exists as a random singular distribution.

Simple way: dim analysis

$$\underbrace{\int |D\phi|^2 d^dx}_{|S|=0} \text{ for } d\le 3$$

$$|S|=0 \Rightarrow |\phi| = \frac{2-d}{2}$$

$$\Rightarrow \phi\in\mathcal{C}^{\frac{2-d}{2}-0} \text{ a.s.}$$

$$d=2 \Rightarrow \phi\in\mathcal{C}^{-0}$$

$$d=3 \Rightarrow \phi\in\mathcal{C}^{-\frac{1}{2}-0}$$

ANOTHER WAY to generate:

How to obtain GIBBS meas. as equilibrium meas.

$$\underbrace{e^{-S(\phi)}}_{\text{function on } \mathbb{R}^\Lambda \text{ configuration space}} d^\Lambda\phi$$

Two observations:

$$\Delta_{\mathbb{R}^\Lambda} \underbrace{1}_{\substack{\text{constant}\\ \text{function}}} = 0 \Rightarrow \underbrace{\left(e^{-S[\phi]} \Delta_{\mathbb{R}^\Lambda} e^{S(\phi)}\right)}_{\text{conjugated operator}} e^{-S(\phi)} = 0$$

$$H = \left(e^{-S(\phi)} \frac{\Delta_{\mathbb{R}^\Lambda}}{2} e^{S(\phi)}\right)^{\dagger} = \frac{\Delta_{\mathbb{R}^\Lambda}}{2} + \underbrace{\nabla S(\phi) \cdot \nabla}_{\text{transport by } \nabla S \cdot \nabla}$$

The $\frac{1}{2}$-group e^{-tH} admits $\dfrac{e^{-S(\phi)}}{\int e^{-S(\phi)}}$ as INVARIANT PROBA MEASURE.

If P has degree $2p \geq 4$, then for $F \in L^1(\mathbb{R}^\Lambda)$ observable: $\forall \phi_0 \in \mathbb{R}^\Lambda$

$$\langle F(\phi) \rangle_\Lambda \overset{\text{déf}}{=} \mathbb{E}_\mu(F) = \lim_{t \to +\infty} \left(e^{-tH} F\right)(\phi_0)$$

Convergence to equilibrium exponentially fast $t \to +\infty$

This means:

$$\left|\mathbb{E}_\mu(F) - (e^{-tH}F)(\phi_0)\right| \lesssim e^{-\kappa t}$$

$(e^{-tH})_{t\in[0,+\infty)}$ MARKOV $\frac{1}{2}$-group associated to diffusion process $(X_t)_t$:

$$dX_t + \overbrace{\nabla S(X_t)}^{\text{transport}}dt = d\vec{B}_t\,, \quad \overset{\text{iid Brownians}}{\vec{B}_t = (B_{t,i})_{i\in\Lambda}}$$

X_t RANDOM VECTOR in $\mathbb{R}^\Lambda$

the relation of X_t with e^{-tH}:

$$e^{-tH}f(\phi_0) = \mathbb{E}\left(f(X_t) \,|\, X_0=\phi_0\right)$$

So to compute correlators $\langle F\rangle$, just consider

X_t solves SDE, $\displaystyle\int_{\mathbb{R}^\Lambda} \mathbb{E}\left(F(X_t)\,|\,X_0=x\right) d^\Lambda x$

$$\Big\downarrow t\to+\infty$$

then let $t\to+\infty$: $\langle F(\phi)\rangle_\Lambda$

QUESTION : ϕ^4_3 constructed in $\mathbb{R}^3$, what about 3-manifold (M,g) ?

BORING PROBLEM

Witten 2112.11614

If a theory exists perturbatively in curved spacetime, and nonperturbatively in flat spacetime, one would expect that it works nonperturbatively in curved spacetime. Unfortunately, not much is available in terms of rigorous theorems, except for special models like two-dimensional conformal field theories. That reflects the general mathematical difficulty of understanding quantum field theory rigorously. One would think that rigorous results for a superrenormalizable theory in curved spacetime might be relatively accessible, but such results are not available.

QFT non perturbative on Mfd:

$P(\phi)_2$	GN_2	ϕ^4_3	Liouville CFT
DIMOCK PICKRELL (2007)	?	?	GUILLARMOU-RHODES-VARGAS (2019, 2022)

GOAL: Try to construct ϕ^4_3 measure on every compact Riemannian (M, g), COVARIANT.

Our tool: Stochastic differential equation on $\mathbb{R}^1$ $\longrightarrow$ SPDE on $\mathcal{D}'(M)$

new config space

$d\vec{B}_t = (dB_{t,i})_{i\in\mathbb{N}} \longrightarrow \mathbb{R}\times M$, $\underbrace{\xi(t,x) \text{ white noise}}_{\text{RANDOM GAUSSIAN } \mathcal{D}'}$

Consider parabolic ϕ^4_3 SPDE:

$$\partial_t \phi + (\Delta_g + 1)\phi = -\lambda \phi^3 + \xi, \quad \lambda \in C^\infty(M)$$

λ coupling function, ϕ^4_3 measure: INVARIANT SPDE meas.

Following Gubinelli-Hofmanova, Jagannath-Perkowski (other approach Hairer)

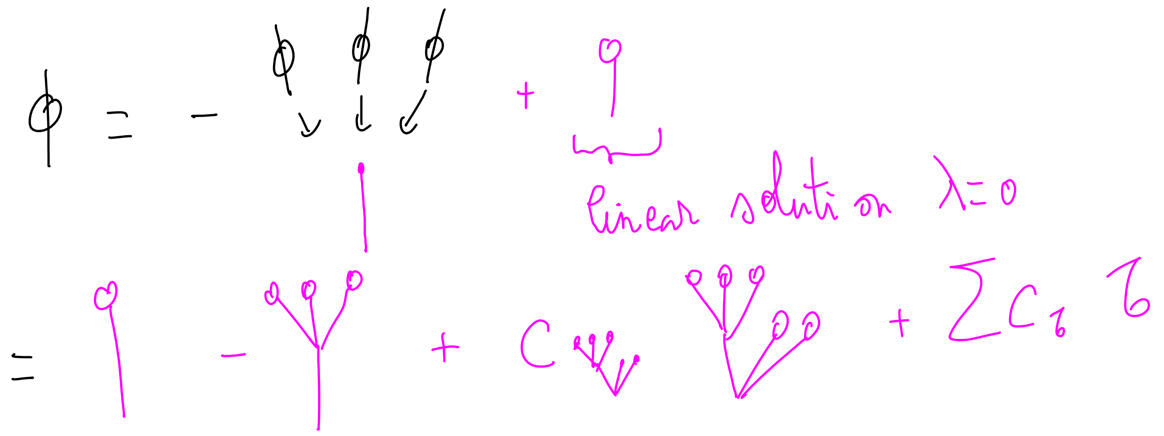
Solve equation graphically. $\mathbb{E}(\xi(t,x)\xi(s,y)) = \delta(t-s)\delta^M(x,y)$

Trees

$\partial_t + (\Delta_g + 1) = \mathcal{L}$ | , ξ o
)

$\mathcal{L}\phi = -\lambda\phi^3 + \circ$

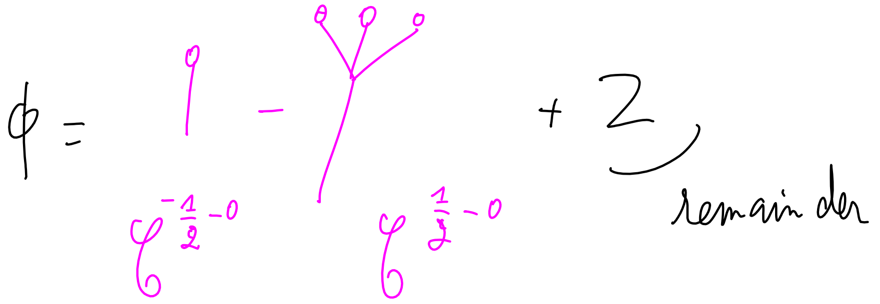
Start writing fixed point equation in terms of trees:



τ : trees. FORMAL EXPANSION

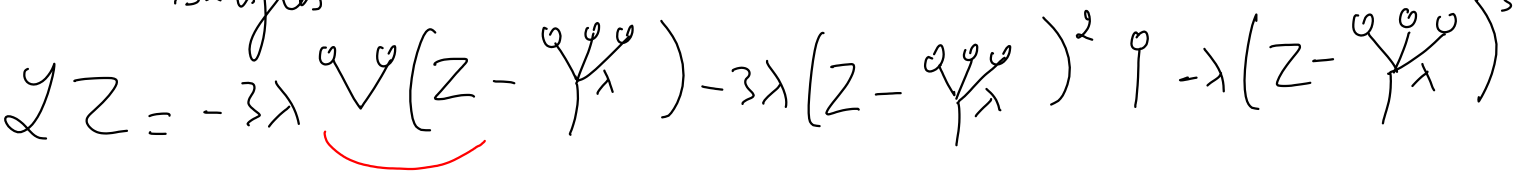
Introduce filtration :

add line | adds +2 to weights



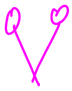
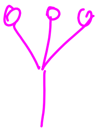
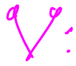
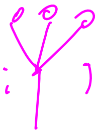
, expect Z + regular

Z satisfies



Thm: $u \in \mathcal{C}^\alpha$, $v \in \mathcal{C}^\beta$, $\alpha+\beta>0$

$$\Rightarrow \quad uv \in \mathcal{C}^{\inf(\alpha,\beta)}$$

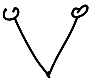
Hence ,  $\xrightarrow{\text{WICK}}$::, ::

by $\xi \to e^{-\varepsilon(\Delta+1)}\xi =: \xi_\varepsilon$ and subtract counterterms:

$$\mathcal{L}\phi + \lambda\phi^3 + 3\lambda C_1 \phi = \xi_\varepsilon$$

$$C_1(\varepsilon) = \frac{1}{8\sqrt{2}\,\pi^{\frac{3}{2}}\sqrt{\varepsilon}}$$

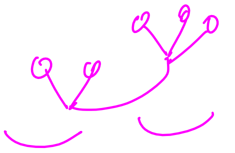
Next problems, need to deal with

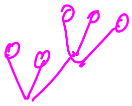
Z expected $\mathcal{C}^{1-}$ BUT $\underset{\mathcal{C}^{1-}}{Z}$  $\mathcal{C}^{-1-}$

$1-0 + -1-0 = -0$

BORDERLINE

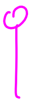
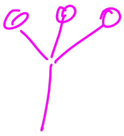
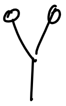
ILL-DEFINED

and  φ^{-1-} $\mathcal{C}^{\frac{1}{2}-}$ needs RENORMALISATION

$R($$) :=$  $- C_2\Big($ $+$ $\Big)$

C_2 new counterterm

Key idea of JP:

$\phi =$  $-$  $+ e^{-3}$ v

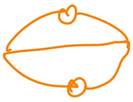
and v NEW UNKNOWN

GOAL of $e^{-3:\mathrm{Y}\lambda:}$ COLE-HOPF transform to kill $\mathrm{V}Z$!

ϑ solves complicated equation, finally

$$\boxed{\mathcal{L}\phi + \left(3\lambda c_1 - 3\lambda^2 c_2\right)\phi + \lambda\phi^3 = \xi}$$

$c_1 = \frac{1}{8\sqrt{2}\,\pi^{\frac{3}{2}}\sqrt{\varepsilon}}$ two counterterms.

$c_2 \approx |\log(\varepsilon)|$ = div part of 

MAIN INGREDIENTS to prove short time existence for F $\mathcal{C}^\infty$

$$\underbrace{F(\mathrm{Y})}_{\mathcal{C}^{1-}} \quad \underbrace{R(\mathrm{V}\mathrm{V})}_{\substack{\text{renorm object} \\ \mathcal{C}^{-1-}}}$$

BORDERLINE product

$1- + -1- = 0-$ ⚠

need to deal with ↑ to do fixed point and solve SPDE.

HARMONIC ANALYSIS

J.M. Bony (1980) + Y. Meyer :

$$\underbrace{u \times v}_{\ll \text{ pointwise product } \gg} = \underbrace{u \succ v} + \underbrace{u \prec v} + \underbrace{u \odot v}_{\text{resonant}}$$

PARAPRODUCTS

$\prec$ paraproducts always well-defined

$$\left(\underset{\mathcal{C}^\alpha}{u}, \underset{\mathcal{C}^\beta}{v}\right) \to u \prec v \in \mathcal{C}^{\inf(\alpha,0)+\beta}$$

$\underset{\mathcal{C}^\alpha}{u} \odot \underset{\mathcal{C}^\beta}{v} \in \mathcal{C}^{\alpha+\beta}$ well-defined only if $\alpha+\beta>0$

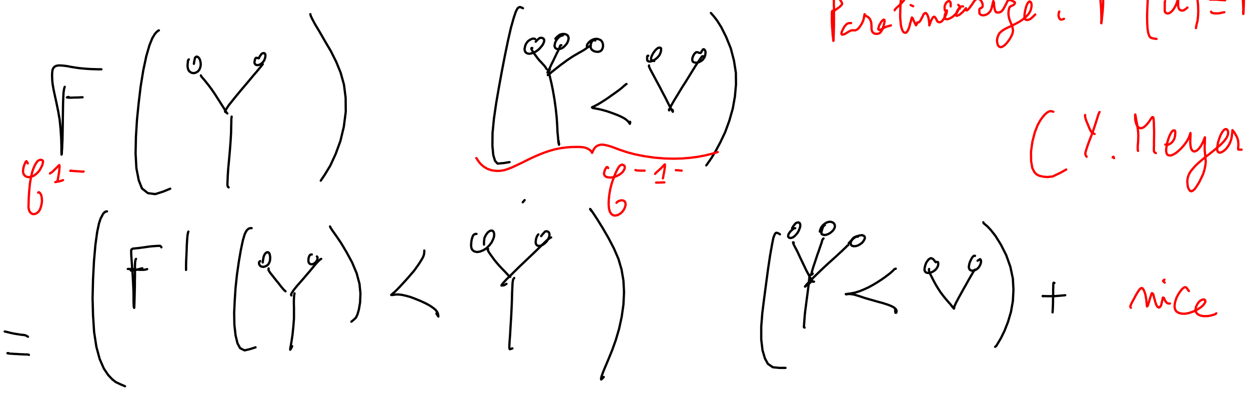
ALL PROBLEMS in $\odot$

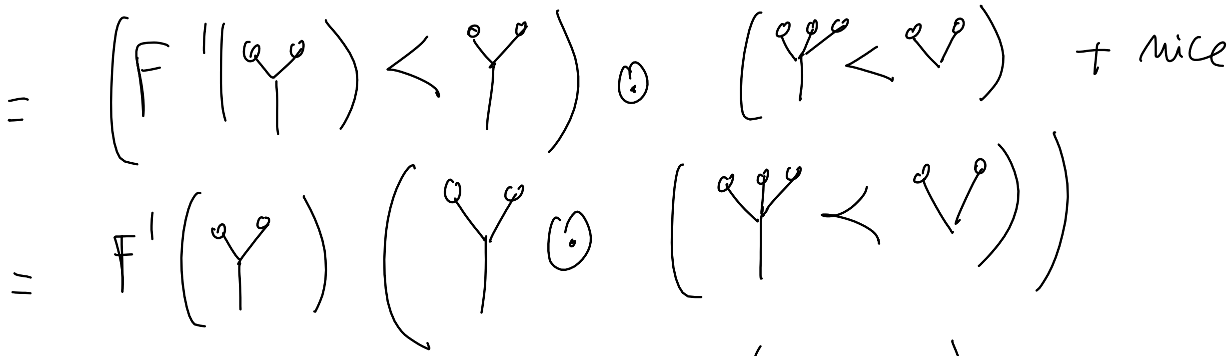
⚠ Operat $\prec$, $\succ$, $\odot$ are non associative

Back to our term, isolate bad:

Paralinearize: $F(u) = F'(u) \prec u$ + nicer

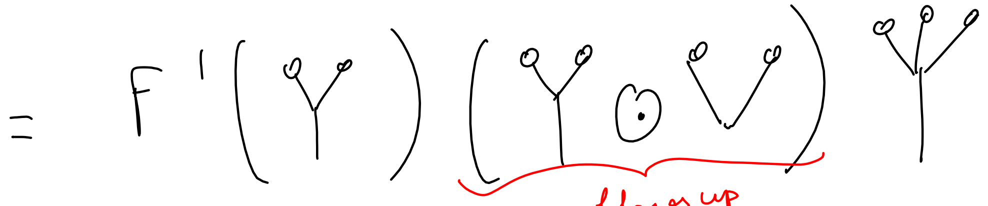
(Y. Meyer, J.M Bony)





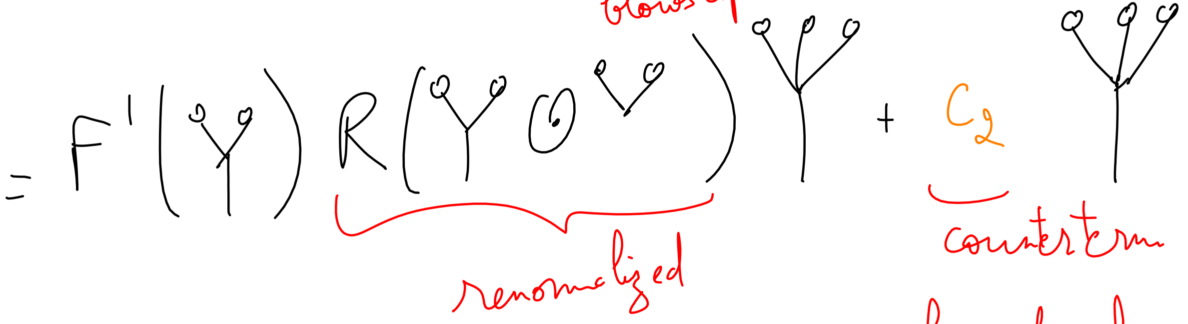
$$+ \text{ term } \underbrace{C(u, v, w) = (u \prec v) \odot w - u (v \odot w)}_{\text{nice} \qquad \text{antisymmetry}}$$

+ nice



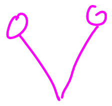
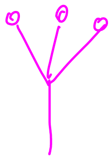
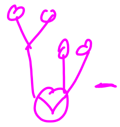
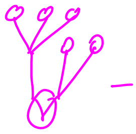
+ nicer commutators

blows up

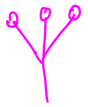
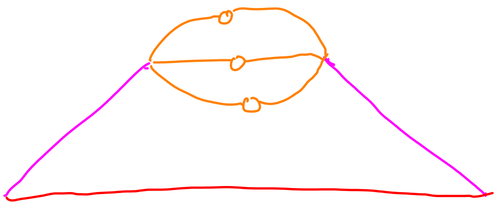


counterterm which is absorbed in the equat.

STOCHASTIC ESTIMATES : prove Regularity

			 $- \frac{C_2}{3}$	$\Vert\nabla$  $\Vert^2 - \frac{C_2}{3}$	 $- C_2$ 
$\mathcal{C}^{-\frac{1}{2}-0}$	$\mathcal{C}^{-1-0}$	$\mathcal{C}^{\frac{1}{2}-0}$	$\mathcal{C}^{-0}$	$\mathcal{C}^{-0}$	$\mathcal{C}^{-\frac{1}{2}-0}$

proof idea : $H^{\frac{1}{2}-}$ Sobolev space

$\mathbb{E}\Big(\Vert$  $\Vert^2_{H^{\frac{1}{2}-}}\Big) =$  $< +\infty$

$$\text{———} = (\Delta+1)^s$$

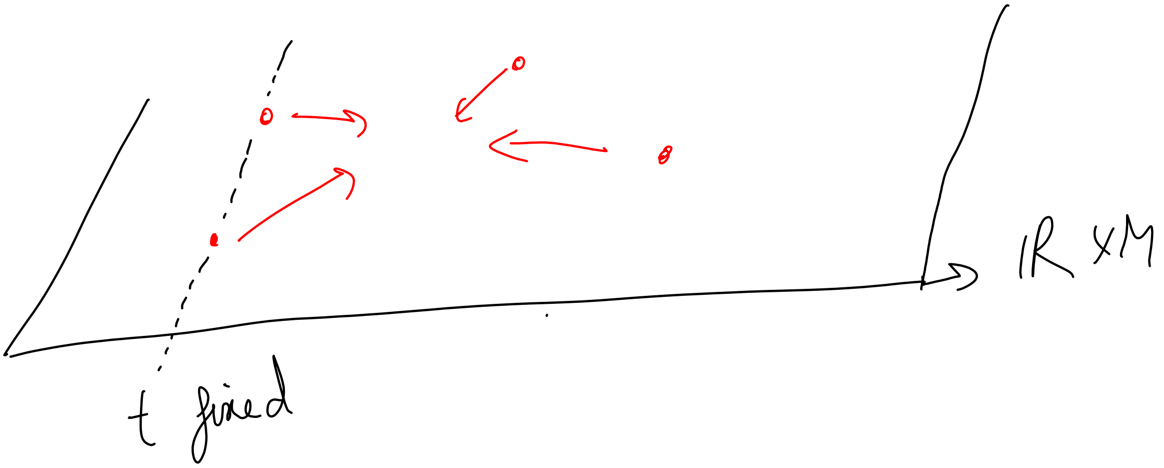
$$\text{—o—} = \frac{e^{-|s_1-s_2|(\Delta+1)}}{2(\Delta+1)}$$

$$\text{———} = e^{-(t-s)(\Delta+1)}$$

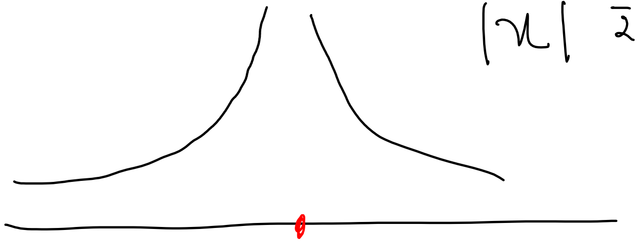
three ≠ propagators

Prove finiteness, Epstein-Glaser :

CONFIG SPACE 4 pts = 4 vertices



TOY EXAMPLE:



$|x|^{-\frac{1}{2}}$ homogeneity $-\frac{1}{2} > -\underset{\underset{-1}{\shortparallel}}{\text{codim}}.$

$\Rightarrow |x|^{-\frac{1}{2}} \in L^1_{loc}$

$\text{codim}\,(\cdot) = 1$

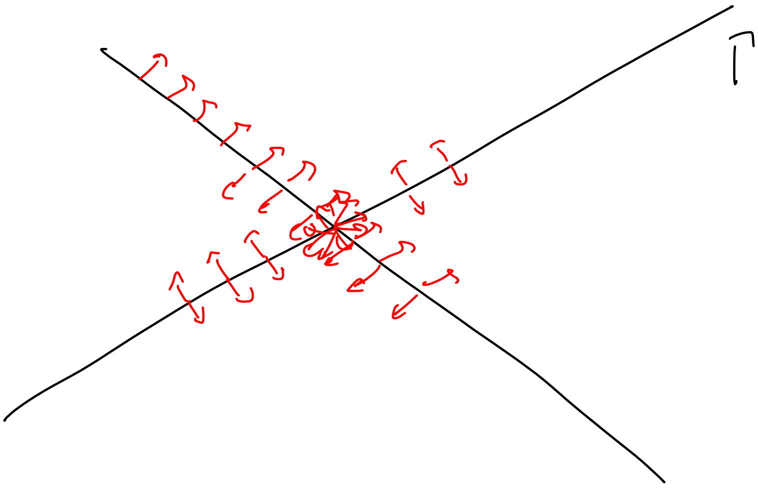
MORALITY: compare homogeneity under SCALING

vs − codim of singularity

Here is the same except

funcθ $\longrightarrow$ $\mathcal{D}'_\Gamma$ distributions whose

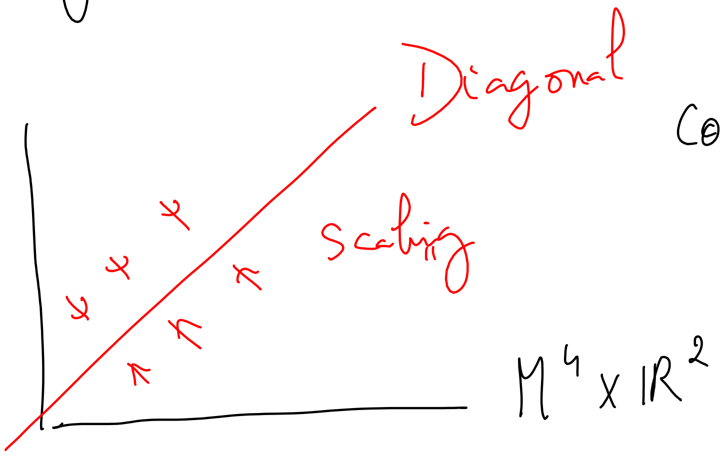
Wave Front set in Γ



Γ = posiθ + codirecθ

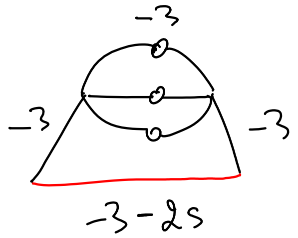
of singularities

Scaling measured with PARABOLIC EULER ρ



$$\operatorname{Codim}_{\text{weighted}} = 3\times 3 + 2\times 2 = 13$$

$t_G =$



$\in \mathcal{D}'_\Gamma(M^4\times\mathbb{R}^2)$

weakly homogeneous $\deg(-3-3-3-3-2s)$

$$(e^{-t\rho} * t_G)\, e^{t(12+2s)} \quad \text{bounded}_{t\geq 0} \text{ in } \mathcal{D}'_\Gamma$$

Refines a concept due to Y. Meyer

Weighted codim $= 13$

$$\underbrace{-13}_{\text{weighted codim}} < \underbrace{-12-2s}_{\text{homogeneity}} \Rightarrow \quad s<\frac{1}{2}$$

THANKS FOR YOUR ATTENTION !