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~~Symmetries & Wess-Zumino consistency condition from the AQFT perspective~~

$\boxed{\text{A new perspective on the WZ const. cond. (joint w/ Brunetti, Dütsch, Fredenhagen)}}$

I. pAQFT (here for n# scalar fields)

M - spacetime, $\mathcal{E} = \mathcal{C}^\infty(M, \mathbb{R}^n)$ field configuration space

$\mathcal{F}$ - functionals $\subset \mathcal{C}^\infty(\mathcal{E}, \mathbb{C})$ w/ some WF set cond. on derivatives

$\mathcal{F}_{loc}$ - local functionals

basic field: $\Phi(x) : \mathcal{E} \to \mathbb{R}$ $\quad \Phi(x)(\varphi) \doteq \varphi(x)$

Lagrangians: $L : \mathcal{D} \to \mathcal{F}_{loc}$, $\quad L(f)[\varphi] = \int_M \mathcal{L}(x)[\varphi] f\, dx$

$\mathcal{D} = \mathcal{C}^\infty_c(M, \mathbb{R})$

e.g. $\mathcal{L}(x) = \frac{1}{2}(\nabla_\mu \varphi \nabla^\mu \varphi + m^2 \varphi^2)(x)$

$\delta L(\psi) \doteq L(f)[\varphi + \psi] - L(f)[\varphi]$ where $f \equiv 1$ on supp $\psi \in \mathcal{E}_c \doteq \mathcal{C}^\infty_c(M, \mathbb{R}^n)$

$\langle dL(\varphi), \psi \rangle \doteq \lim_{t \to 0} \frac{1}{t} L(t\psi)$ for free theory $dL(\varphi) = P\varphi$

P has retarded / advanced Green functions $\Delta^{R/A}$ (glob. hyp)

$\Delta = \Delta^R - \Delta^A$ and $\{F, G\} \doteq \langle F^{(1)}, \Delta G^{(1)} \rangle$ Peierls bracket, $F, G \in \mathcal{F}_{loc}$

$\Delta^+ = \frac{i}{2}\Delta + H$ s.t. Δ^+ satisfies appropriate WF-set condition

(H symmetric)

$F \star G \doteq m \circ e^{\hbar \langle \Delta^+, \frac{\delta}{\delta\varphi} \otimes \frac{\delta}{\delta\varphi} \rangle} F \otimes G$, $\quad F, G \in \mathcal{F}_{reg}$

$\Delta^F = \frac{i}{2}(\Delta^R + \Delta^A) + H$, $\quad F \cdot_T G \doteq m \circ e^{\hbar \langle \Delta^F, \frac{\delta}{\delta\varphi} \otimes \frac{\delta}{\delta\varphi} \rangle} = J(J^{-1}F \cdot J^{-1}G)$

$F, G \in \mathcal{F}_{reg}[[\hbar]]$

$= \begin{cases} F \star G & F \gtrsim G \\ G \star F & G \gtrsim F \end{cases}$

$J = e^{\frac{\hbar}{2} \langle \Delta^F, \frac{\delta^2}{\delta\varphi^2} \rangle}$

Formally: $JF(\varphi) = \int F(\varphi - \phi)\, d\mu(\phi)$ $\quad$ gaussian measure w/ cov. Δ^F

Renormalisation: extend T to $\mathcal{F}_{loc}$ and $\mathcal{J}$ to $\mathcal{F}_{\otimes loc} = \bigoplus_{k=0}^{\infty} \mathcal{F}_{k,loc}$ (Epstein-Glaser)

set $T|_{\mathcal{F}_{loc}} = id$ — k-fold prod. of $\mathcal{F}_{loc}$

s.t. $T_n(F_1, \dots, F_n) = T_k(F_1, \dots, F_k) \star T_{n-k}(F_{k+1}, \dots, F_n)$

for $F_1, \dots, F_k \gtrsim F_{k+1}, \dots F_n$

formal S-matrix: $\mathcal{S}(F) \doteq e_T^{\frac{i}{\hbar}F} = T e^{\frac{i}{\hbar}F}$

(S1) $\mathcal{S}(F+H+G) = \mathcal{S}(F+H)\mathcal{S}(H)^{-1}\mathcal{S}(H+G)$ ~~$\mathcal{S}(F)\star\mathcal{S}(G)$~~ if $\operatorname{supp} F \cap J_-(\operatorname{supp} G) = \emptyset$

(S2) $\mathcal{S}^{(\delta L(\psi))}(F) = \mathcal{S}(F^\psi + \delta L(\psi))$, $F^\psi[\varphi] = F[\varphi + \psi]$

$\delta L(\psi) = L(f)^\psi - L(f)$ $\qquad Z \in \mathcal{R}_0$

Renorm. group: $\mathcal{R}_0$ relate different possible T, i.e. $\tilde{\mathcal{S}}(F) = \mathcal{S}(Z(F))$

Symmetries: $\mathcal{G}_c$ - compactly-supported automorphism of the affine bundle $M \times \mathbb{R}^n$

↳ point diffeos $\varrho: M \to M$

↳ affine field redefinitions

$g \in \mathcal{G}_c$ acts on functionals: $g_* F(\varphi) = F(g(\varphi))$

$\delta_g L \doteq g_* L(f) - L(f)$ $\quad f \equiv 1$ on $\operatorname{supp} g$

$g_L F \doteq \delta_g L + g_* F$ $\rightsquigarrow$ relate to renorm. group $\mathcal{R}_0$

off-shell UAMWI: $L_q \doteq L - \langle \Phi, q \rangle$ — source $q \in \mathcal{E}_{dens}(M, \mathbb{R}^n)$

$\exists$ "anomaly map" $\zeta: \mathcal{G}_c \to \mathcal{R}$ s.t. $\zeta_e = id_{\mathcal{F}_{loc}}$

$\operatorname{supp} \zeta_g \subset \operatorname{supp} g$

satisfying: $\zeta_{gh} = \zeta_h h_L^{-1} \zeta_g h_L$ $\quad g, h \in \mathcal{G}_c$

s.t. $\forall q \in \mathcal{E}_{dens}(M, \mathbb{R}^n)$: $\left(\mathcal{S} \circ g_{L_q}(F) = \mathcal{S} \circ \zeta_g(F)\right)\Big|_{\phi \text{ s.t. } \frac{\delta L}{\delta \phi}[\phi] = q}$

③

$\underset{\cup}{\mathrm{Lie}\,\mathcal{R}}$

Infinitesimal: Δ: Lie $\mathcal{G}_c \ni X \mapsto \Delta X \doteq \frac{d}{d\lambda}\Big|_{\lambda=0} S_{g\lambda}$

AMWI: $e_T^{iF} \cdot_T (\partial_X F + \partial_L L - \Delta X(F)) = \int (e_T^{iF} \cdot_T (\partial_X \Phi(x)) \frac{\delta L}{\delta \varphi(x)}$

II. Wess-Zumino consisteny relation

$\mathcal{G}_0 \subset \mathcal{G}_c$ orthogonal field redefinitions $g \in \mathcal{C}^\infty(M, SO(n,\mathbb{R}))$

↳ gauge transf.

$A = g^{-1} dg$ ⇝ external gauge field (background connection) so$(n,\mathbb{R})$-valued 1-form

$L_A = \frac{1}{2}\langle (d+A)\varphi, (d+A)\varphi \rangle$

$(g_* L_A)(\varphi) = L_{A^g}(\varphi)$, where $A^g = g(dg^{-1}) + gAg^{-1}$

$V(A) \doteq \int (L_A - L)$ interaction induced by A

Effective action:

$W(j) = S(V(A) + \langle \Phi, j \rangle)[\varphi = 0]$

$\Gamma(A, \phi) = \langle \phi, J \rangle - i \log\Big(S(V(A) + \langle \Phi, J \rangle)[\varphi = 0]\Big)$, $\phi \in \mathcal{E}_c(M, \mathbb{R}^n)$

$J(\phi, A)$ solves: $\phi = -i \frac{\delta}{\delta j} \log\Big(S(V(A) + \langle \Phi, j \rangle)[\varphi = 0]\Big)\Big|_{j=0}$

↳ $J = \Box_A \phi$ ↗ d'Alembertian for an external gauge field A

$\Gamma(A, \phi) = \int L_A(\phi) - i \log\Big(S(V(A) + \langle \Phi, \Box_A \phi \rangle)[\varphi = 0]\Big)$

↳ gauge transformations: gauge inv. if no anomalies

$X \mapsto \partial_X^{A,\phi}$ infinitesimal gauge transf. $\begin{pmatrix} A \mapsto A^g \\ \phi \mapsto g\phi \end{pmatrix}$

$\boxed{G(X, A) \doteq \partial_X^{A,\phi} \Gamma(A, \phi)}$ anomaly

Wess-Zumino consisteny relation:

$$\boxed{\partial_X^A G(Y, A) - \partial_Y^A G(X, A) = G([X, Y], A)}$$

Derivation using UAMWI:

$$\underbrace{g_L(V(A)+\langle\Phi,\Box_A\varphi\rangle)}=\underbrace{V(A^g)+\langle\Phi,\Box_{A^g}(g\phi)\rangle}$$

$$\text{UAMWI} \Rightarrow S(\quad)|_{\phi=0}=S(\zeta_g(\quad))|_{\varphi=0}$$

Fact: $\zeta_g(V(A)+\langle\Phi,\Box_A\phi\rangle)=\zeta_g(V(A))+\langle\Phi,\Box_A\phi\rangle$ φ-indep.

$$=V(A)+\langle\Phi,\Box_A\phi\rangle+\underbrace{G(g,A)}_{\text{local!}}$$

$$\Rightarrow S(V(A^g)+\langle\Phi,\Box_{A^g}(g\phi)\rangle)|_{\varphi=0}=S(V(A)+\langle\Phi,\Box_A\phi\rangle)|_{\phi=0}\,e^{i\,G}$$

$$\Rightarrow \Gamma(A^g,g\phi)=\Gamma(A,\phi)+G(g,A)$$

$$\Rightarrow G(gh,A)=G(g,A^h)+G(h,A^g)$$, set $g=e^{-\lambda X}$, $h=e^{-\mu Y}$ and differentiate

$$\Rightarrow \text{WZ with } G(X,A)=\frac{d}{d\lambda}\Big|_{\lambda=0}G(e^{-\lambda X},A)$$

III. Generalized Wess-Zumino consistency

ζ intertwines two actions of $\mathcal{G}_c$ on $\mathcal{F}_{loc}$

$(g,F)\mapsto g_L F$ and $(g,F)\mapsto g_L\zeta_g^{-1}(F)$

$\Rightarrow \Delta$ Lie-algebraic cocycle satisfying:

$$\boxed{\Delta([X,Y])(F)=-[\Delta X,\Delta Y]_{\mathrm{Lie}\,\mathcal{R}_c}(F)+(\tilde{\partial}_X\Delta Y)(F)-(\tilde{\partial}_Y\Delta X)(F)}$$

$X,Y\in\mathrm{Lie}\,\mathcal{G}_c$ generalised WZ consistency

where the action of $\mathcal{G}_c$ on $\mathcal{R}_c$ is defined by:

$$(\tilde{\partial}_X Z)(F)=\partial_X(Z(F))-\langle Z'(F),\partial_X(F+L)\rangle$$

IV. Relation to BV

$$\Delta X(F) = i\,\underline{\Delta}_F\, \partial_X$$

↳ renormalised BV Laplacian

~~?~~ "bare" BV Laplacian acts on $\Lambda\Gamma(T\mathcal{E})$

$$\underline{\Delta} = \int \frac{\delta^2}{\delta\varphi(x)\,\delta\varphi^{\ddagger}(x)} \qquad \varphi^{\ddagger}(x) \equiv \frac{\delta}{\delta\varphi(x)} \text{ generators of the fibre}$$

$$\underline{\Delta}(X \wedge Y) = (\underline{\Delta} X)Y + X(\underline{\Delta} Y) + \{X, Y\}$$

↳ BV bracket

also for $\underline{\Delta}_F$ if X, Y affine in φ.

$$\underline{\Delta}^2 = 0 \Rightarrow \partial_X(\underline{\Delta} Y) - \partial_Y(\underline{\Delta} X) = \underline{\Delta}(\{X, Y\})$$

For the renormalised case: $\hat{s} = s - i\,\underline{\Delta}_F$

("classical" BV over s; ↳ quantum BV)

$$\hat{s}^2 = 0 \Rightarrow \underline{\Delta}_F(\{X, Y\}) = \Delta([X, Y])(F)$$

$$= \underbrace{\langle(\Delta Y)'(F), \Delta X(F)\rangle - \langle(\Delta X)'(F), \Delta Y(F)\rangle}_{[\Delta X, \Delta Y]_{\text{Lie}}}$$

$$+ \partial_X(\Delta Y(F)) - \partial_Y(\Delta X(F))$$

$$- \langle(\Delta Y)'(F), \partial_X(L+F)\rangle + \langle(\Delta X)'(F), \partial_Y(L+F)\rangle$$