2-MONOIDAL CATEGORIES IN QFT

(ONGOING WORK)

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§1. MOTIVATION:

- GIVEN A SPACETIME M & VECTOR BUNDLE E OVER M, A PROPAGATOR IS

ISO → "THIS SHOULD BE A BIALG" — "THIS SHOULD BE AN ALG"

$$\Delta \in \mathrm{Hom}_{C^\infty(M)\otimes C^\infty(M)}\left(\Gamma(J^\cdot E)\otimes\Gamma(J^\cdot E),\ \mathcal{D}(M)\otimes\mathcal{D}(M)\right)$$

JETS OF ORDER ⋅ — DISTRIBUTIONS

(*)
- WE WANT TO EXTEND Δ TO A LAPLACE PAIRING ON SOMETHING LIKE $T(S_{C^\infty(M)}\Gamma(J^\cdot E))$ [BORCHERDS, BROUDER-FAUSER-FRABETTI-OECKL]
THIS IS A BIALGEBRA IN A 2-MONOIDAL CATEGORY.

RECALL DEFINITION OF LAPLACE PAIRING

OUR AIM IS TO PRESENT THESE ALGEBRAIC CONSTRUCTIONS IN A MORE GENERAL SETTING OF FUNCTORS, AND SHOW HOW THE STEPS IN (*) ARE A PARTICULAR CASE.

§2. 2-MONOIDAL CATEGORIES (OR DUOIDAL CATEGORIES)

[AGUIAR-MAHAJAN '10; BATANIN-MARKL '12; STREET '12]

DEF A 2-MONOIDAL CATEGORY $(\mathcal{C},\otimes,I_\otimes,\boxtimes,I_\boxtimes,sh)$ IS A CATEGORY $\mathcal{C}$ WITH TWO MONOIDAL STRUCTURES $(\mathcal{C},\otimes,I_\otimes)$ AND $(\mathcal{C},\boxtimes,I_\boxtimes)$ TOGETHER WITH

$$(A\otimes B)\boxtimes(C\otimes D)\xrightarrow{sh}(A\boxtimes C)\otimes(B\boxtimes D) \qquad \text{[INTERCHANGE LAW]}$$

AND

$$I_\otimes\boxtimes I_\otimes\xrightarrow{\mu}I_\otimes\xleftarrow{\nu}I_\boxtimes\xrightarrow{\Delta}I_\boxtimes\otimes I_\boxtimes$$

S.TH

(1) COHERENCE AXIOMS ON ASSOCIATIVITY [TWO HEXAGONS]

(2) COMPATIBILITY AXIOMS BETWEEN sh AND THE UNITS $I_\boxtimes$ AND $I_\otimes$ [FOUR SQUARES]

(3) $(I_\otimes,\mu,\nu)$ IS A (UNITARY) ALG IN $(\mathcal{C},\boxtimes,I_\boxtimes)$
$(I_\boxtimes,\Delta,\nu)$ IS A (COUNITARY) COALG IN $(\mathcal{C},\otimes,I_\otimes)$

EQUIV TO EITHER
$\otimes:(\mathcal{C},\boxtimes,I_\boxtimes)^{\times 2}\to(\mathcal{C},\boxtimes,I_\boxtimes)$ & $I_\otimes: e\to(\mathcal{C},\boxtimes,I_\boxtimes)$ ARE (LAX) MONOIDAL
OR
$\boxtimes:(\mathcal{C},\otimes,I_\otimes)^{\times 2}\to(\mathcal{C},\otimes,I_\otimes)$ & $I_\boxtimes: e\to(\mathcal{C},\boxtimes,I_\boxtimes)$ ARE OPLAX MONOIDAL

RK 2-MON CAT EQUIVALENT TO PSEUDOMONOID IN THE MONOIDAL 2-CAT $\mathcal{L}(CAT)$.

ADDENDUM TO DEF | A 2-MON CAT is $\boxed{\text{SYMMETRIC}}$ if $\otimes$ & $\boxtimes$ ARE SYMMETRIC, THEY ARE COMPATIBLE WITH sh (TWO SQUARES), $(I_\otimes, \mu, \nu)$ is A COMM. ALG & $(I_\boxtimes, \Delta, \nu)$ is A COCOMM. COALG.

EXAMPLE 0 | A SYMMETRIC CAT $(\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}}, \tau^{\mathcal{C}})$ [OR EVEN JUST BRAIDED!] GIVES A SYMMETRIC 2-MON CAT [OR BRAIDED IN THAT CASE FOR $\mathcal{C}$] WITH $\otimes = \boxtimes = \otimes_{\mathcal{C}}$, $I_\otimes = I_\boxtimes = I_{\mathcal{C}}$ AND $sh = id \otimes_{\mathcal{C}} \tau^{\mathcal{C}} \otimes_{\mathcal{C}} id$.

EXAMPLE | TAKE

• $\boxed{(\mathcal{G}, \overset{\otimes_{\mathcal{G}}}{\sqcup}, \underset{I_{\mathcal{G}}}{\emptyset})}$ = SYMMETRIC MONOIDAL CAT WITH OBJECTS IN $\mathbb{N}_0$, DISCRETE, $n \sqcup_{\mathcal{G}} m = n+m$, $\emptyset = 0$ [MORE GENERALLY, WE CAN TAKE $\mathcal{G}$ A SUBPROP OF $(FinSet, \sqcup, \emptyset)$]

• $\boxed{(\mathcal{E}, \otimes_{\mathcal{E}}, I_{\mathcal{E}}, \tau^{\mathcal{E}})}$ = SYMMETRIC $\mathbb{C}$-LINEAR MONOIDAL CAT, ASSUMED TO BE COMPLETE AND COCOMPLETE WITH TENSOR PRODUCTS COMMUTING WITH COLIMITS [IN QFT, $\mathcal{E}$ WILL BE CAT $CLCS_{HD}$, $\otimes_{\mathcal{E}} = \tilde{\otimes}_\beta$ CONVENIENT TENSOR PRODUCT.]

THEN, SET $Fun(\mathcal{G}, \mathcal{E}) :=$ CAT OF FUNCTORS FROM $\mathcal{G}$ TO $\mathcal{E}$ = $\mathbb{N}_0$-GRADED OBJECTS IN $\mathcal{E}$

WITH $\otimes_C : Fun(\mathcal{G}, \mathcal{E})^2 \to Fun(\mathcal{G}, \mathcal{E})$

$$\begin{cases} \boxed{(F \otimes_C G)(I) := \bigoplus_{I = I_1 \sqcup I_2} F(I_1) \otimes_{\mathcal{E}} F(I_2)} \ \& \ (F \otimes_C G)(f) := \bigoplus_{I = I_1 \sqcup I_2} F(f|_{I_1}) \otimes_{\mathcal{E}} G(f|_{I_2}) \\ (t \otimes_C u)(I) := \bigoplus_{I = I_1 \sqcup I_2} t(I_1) \otimes_{\mathcal{E}} u(I_2) \end{cases}$$

[CAUCHY TENSOR PRODUCT]

[WITH OR WITHOUT op]

& $\otimes_H : Fun(\mathcal{G}^{op}, \mathcal{E})^2 \to Fun(\mathcal{G}^{op}, \mathcal{E})$

$$\begin{cases} \boxed{(F \otimes_H G)(I) := F(I) \otimes_{\mathcal{E}} G(I)} \ \& \ (F \otimes_H G)(f) := F(f) \otimes_{\mathcal{E}} G(f) \\ (t \otimes_H u)(I) := t(I) \otimes_{\mathcal{E}} u(I) \end{cases}$$

FOR $F, G \in Fun(\mathcal{G}, \mathcal{E})$ AND $t : F \to F'$, $u : G \to G'$ NAT TRANSF.
F', G'

LET $I_H \in Fun(\mathcal{G}, \mathcal{E})$ BE $\boxed{I_H(I) = I_{\mathcal{E}}, \forall I \in \mathcal{G}}$

$I_C \in Fun(\mathcal{G}, \mathcal{E})$ BE $\boxed{I_C(I) = \begin{cases} I_{\mathcal{E}} & \text{if } I = \emptyset \\ 0 & \text{if } I \neq \emptyset \end{cases}}$ (ZERO OF $\mathcal{E}$)

$\tau^H(F, G) : F \otimes_H G \to G \otimes_H F$ & $\tau^C(F, G) : F \otimes_C G \to G \otimes_C F$

$\tau^H(F, G)(I) = \tau^{\mathcal{E}}(F(I), G(I))$; $\tau^C(F, G)(I) = \bigoplus_{I = I_1 \sqcup I_2} \tau^{\mathcal{E}}(F(I_1), G(I_2))$

LET

$$sh_{CH}(F_1,F_2,G_1,G_2): (F_1 \otimes_C F_2) \otimes_H (G_1 \otimes_C G_2) \longrightarrow (F_1 \otimes_H G_1) \otimes_C (F_2 \otimes_H G_2)$$

EVAL AT $I \in \mathcal{G}$ BE

$$\bigoplus_{I_1 \sqcup I_2 = I} (F_1(I_1) \otimes_{\mathcal{E}} F_2(I_2)) \otimes_{\mathcal{E}} \bigoplus_{I_1' \sqcup I_2' = I} (G_1(I_1') \otimes_{\mathcal{E}} G_2(I_2')) \longrightarrow \bigoplus_{I_1 \sqcup I_2 = I} (F_1(I_1) \otimes_{\mathcal{E}} G_1(I_1)) \otimes_{\mathcal{E}} (F_2(I_2) \otimes_{\mathcal{E}} G_2(I_2))$$

$\downarrow$ PROJ TO $I_1 = I_1'$ & $I_2 = I_2'$ $\circlearrowleft$

$$\bigoplus_{I_1 \sqcup I_2 = I} (F_1(I_1) \otimes_{\mathcal{E}} F_2(I_2)) \otimes_{\mathcal{E}} (G_1(I_1) \otimes_{\mathcal{E}} G_2(I_2)) \xrightarrow{\bigoplus_{I_1 \sqcup I_2 = I} id_{F_1(I_1)} \otimes \tau^{\mathcal{E}}(F_2(I_2), G_1(I_1)) \otimes_{\mathcal{E}} id_{G_2(I_2)}}$$

THEN [CF. [AM], EX. 6.22 & PROP 8.68]

$\boxed{\text{FACT}}$ $(Fun(\mathcal{G},\mathcal{E}), \otimes_C, I_C, \otimes_H, I_H, sh_{CH})$ IS A SYMMETRIC 2-MON CAT.

$\boxed{\text{NOTATION:}}$ $\boxed{Fun_{CH}(\mathcal{G},\mathcal{E})}$

WITH APPROPRIATE $\mu_H: I_C \otimes_H I_C \to I_C$, $\Delta_C: I_H \to I_H \otimes_C I_H$, $\nu_{HC}: I_H \to I_C$

E.G. $\mu_H(I) = \begin{cases} l_{I_{\mathcal{E}}} & \text{IF } I = \emptyset \\ 0 & \text{IF } I \neq \emptyset \end{cases}$, $\nu_{HC}(I) = \begin{cases} id_{I_{\mathcal{E}}} & \text{IF } I = \emptyset \\ 0 & \text{IF } I \neq \emptyset \end{cases}$

AND

$\boxed{\text{OBVIOUS ONE}}$

$\boxed{\text{FACT}}$ $(Fun(\mathcal{G},\mathcal{E}), \otimes_H, I_H, \otimes_C, I_C, sh_{HC})$ IS A SYMMETRIC 2-MON CAT.

$\boxed{\text{NOTATION:}}$ $\boxed{Fun_{HC}(\mathcal{G},\mathcal{E})}$

$\boxed{\text{NOTATION}}$ $Fun_C(\mathcal{G},\mathcal{E}) := (Fun(\mathcal{G},\mathcal{E}), \otimes_C, I_C, \tau^C)$
$Fun_H(\mathcal{G},\mathcal{E}) := (Fun(\mathcal{G},\mathcal{E}), \otimes_H, I_H, \tau^H)$ SYM. MON. CAT.

§3. PARTIAL ALGEBRAIC STRUCTURES

$\boxed{\text{TECHNICAL ASSUMPTION: WITH IMAGES}}$

$\boxed{\text{DEF}}$ [CF. BORCHERS '72] A $\boxed{\text{PARTIAL ALGEBRA}}$ IN $(\mathcal{E}, \otimes_{\mathcal{E}}, I_{\mathcal{E}})$ IS A FAMILY OF OBJECTS $(A_n)_{n \in \mathbb{N}}$ WITH $A_n \overset{\boxed{\text{SUBOBJECT}}}{\subseteq} A_1^{\otimes_{\mathcal{E}} n}$ S.TH IT FACTORS

$$\underbrace{\underset{\boxed{\text{SUBOBJECT}}}{A_n \subseteq} \bigcap_{i=0}^{n-2} A_1^{\otimes_{\mathcal{E}} i} \otimes_{\mathcal{E}} A_2 \otimes_{\mathcal{E}} A_1^{\otimes_{\mathcal{E}}(n-i-2)}}_{\boxed{\text{PULL-BACK}}} \xrightarrow{can} A_1^{\otimes_{\mathcal{E}} n}, \quad \forall n \geq 2,$$

WITH MAP $\mu: A_2 \longrightarrow A_1$ S.TH

$$Img\left(id_{A_1^{\otimes_{\mathcal{E}} i}} \otimes_{\mathcal{E}} \mu \otimes_{\mathcal{E}} id_{A_1^{\otimes_{\mathcal{E}}(n-i-2)}}\Big|_{A_n} : A_n \longrightarrow A_1^{\otimes_{\mathcal{E}}(n-1)} \right)$$

FACTORS THROUGH $A_{n-1} \xrightarrow{can} A_1^{\otimes_{\mathcal{E}}(n-1)}$, $\forall n \geq 3$, $i \in [\![0, n-2]\!]$, AND

$\mu \circ (id_{A_1} \otimes_{\mathcal{E}} \quad) = \quad \circ (\mu \otimes_{\mathcal{E}} id_{A_1})$.

ADDENDUM TO DEF IF $\mathcal{C}$ SYMMETRIC, A PARTIAL ALG IS COMM. IF

$\mu \circ \tau^{\mathcal{C}}(A_1, A_1) = \mu$. ($\Rightarrow \tau^{\mathcal{C}}(A_1,A_1)(A_2) = A_2$?).

EXAMPLE/PROP M MANIFOLD (E.G. SPACETIME), $\mathcal{D}'(M)$ HAS A STRUCTURE OF COMMUTATIVE PARTIAL ALGEBRA WITH $A_1 = \mathcal{D}'(M)$, $A_n = \{\omega \in \mathcal{D}'(M^n) \mid WF(\omega) \cap N_{\Delta_n} = \emptyset\}$

WHERE $N_{\Delta_n} = \{(p_1, \dots, p_n, v_1, \dots, v_n) \in (T^*M)^n \setminus 0 \mid \sum_{i \in I} v_i \neq 0, \forall I \subseteq \{1,\dots,n\} \text{ s.th. } p_i = p_j \forall i,j \in I\}$

RK SIMILAR DEFINITION FOR MODULES, MORPHISMS OF ALG/MOD, ... (WE COULD ALSO PICK SOMETHING LIKE A CONIC CONDITION ?)

§4. BIALGEBRAS IN 2-MONOIDAL CATEGORIES

[AGUIAR-MAHAJAN, '10]

DEF A BIALGEBRA (OR THE DOMAIN OF OUR LAPLACE PAIRING TO BE) IN A 2-MON CAT $(\mathcal{C}, \otimes, I_\otimes, \boxtimes, I_\boxtimes, sh)$ IS AN OBJECT B WITH

- AN ALG STRUCTURE (B, μ_B, η_B) IN $(\mathcal{C}, \boxtimes, I_\boxtimes)$
- A COALG STRUCTURE $(B, \Delta_B, \epsilon_B)$ IN $(\mathcal{C}, \otimes, I_\otimes)$

S.TH

$$\begin{tikzcd} & B \arrow[dr, "\Delta_B"] & \\ B\boxtimes B \arrow[ur, "\mu_B"] \arrow[d, "\Delta_B \boxtimes \Delta_B"'] & \circlearrowleft & B\otimes B \\ (B\otimes B)\boxtimes(B\otimes B) \arrow[rr, "sh"] & & (B\boxtimes B)\otimes(B\boxtimes B) \arrow[u, "\mu_B\otimes\mu_B"'] \end{tikzcd}
\qquad
\begin{tikzcd} B\boxtimes B \arrow[r, "\mu_B"] \arrow[d, "\epsilon_B\boxtimes\epsilon_B"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, "\epsilon_B"] \\ I_\otimes\boxtimes I_\otimes \arrow[r, "\mu"] & I_\otimes \end{tikzcd}
\qquad
\begin{tikzcd} B \arrow[r, "\Delta_B"] \arrow[dr, phantom, "\circlearrowleft"] & B\otimes B \\ I_\boxtimes \arrow[u, "\eta_B"] \arrow[r, "\Delta"] & I_\boxtimes\otimes I_\boxtimes \arrow[u, "\eta_B\otimes\eta_B"'] \end{tikzcd}
\qquad
\begin{tikzcd} I_\boxtimes \arrow[rr, "\sim"] \arrow[dr, "\eta_B"'] & \circlearrowleft & I_\otimes \\ & B \arrow[ur, "\epsilon_B"'] & \end{tikzcd}$$

EXAMPLE/PROP [BASED ON BORCHERDS '01]

A COMM ALG IN $(\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}}, \tau^{\mathcal{C}})$

SET $T_*(A) \in Fun(\mathcal{F}, \mathcal{C})$

BY $T_*(A)(I) = \bigotimes_{i\in I} A$, $T_*(A)(f)\left(\bigotimes_{i\in I} x_i\right) = \bigotimes_{j\in J} y_j$, $y_j := \mu^{[\# f^{-1}(\{j\})]}\left(\bigotimes_{i\in f^{-1}(\{j\})} x_i\right)$ $(f : I \to J)$

THEN $T_*(A)$ IS COMM. ALG IN $Fun_H(\mathcal{F}, \mathcal{C})$.

IT IS ALSO A BIALGEBRA (COMM & COCOMM) IN THE SYMM. 2-MON CAT $Fun_{CH}(\mathcal{F}, \mathcal{C})$.

($\epsilon_{T_*(A)} : T_*(A) \to I_C$ WITH $\epsilon_{T_*(A)}(I) = \begin{cases} id_{I_{\mathcal{C}}} & \text{IF } I = \emptyset \\ 0 & \text{IF } I \neq \emptyset \end{cases}$, $\Delta_{T_*(A)} : T_*(A) \to T_*(A) \otimes_C T_*(A)$)

WITH ($\mathcal{I} = \mathcal{I}_1 \sqcup \mathcal{I}_2$)

$$\begin{tikzcd}
T_\circledast(A)(\mathcal{I}) \arrow[r, "\Delta_{T_\circledast(A)}(\mathcal{I})"] \arrow[d, equal] & (T_\circledast(A)\otimes_C T_\circledast(A))(\mathcal{I}) \arrow[r, "\text{CAN}", "\text{PROJ}"'] & T_\circledast(A)(\mathcal{I}_1)\otimes_{\mathcal{C}} T_\circledast(A)(\mathcal{I}_2) \arrow[d, equal] \\
\bigotimes_{i\in\mathcal{I}} A \arrow[rr, dashed, "\text{CAN ISO GIVEN BY } \tau^{\mathcal{C}}"] & & \bigotimes_{i\in\mathcal{I}_1} A \otimes_{\mathcal{C}} \bigotimes_{i\in\mathcal{I}_2} A
\end{tikzcd}$$

$\boxed{\text{FOR QFT: } A = C^\infty(M)}$

§5. MODULES OVER BIALGEBRAS IN 2-MON CAT

LET $\mathcal{M} = \mathrm{Mod}_{T_\circledast(A)}(\mathcal{F}, \mathcal{C})$ = MODULES OVER $T_\circledast(A)$ IN $\mathrm{Fun}_H(\mathcal{F}, \mathcal{C})$

THEN $(\mathcal{M}, \underbrace{\otimes_C}_{\boxtimes_{\mathcal{M}}}, \underbrace{I_C}_{I_\boxtimes})$ IS SYMM. MON CAT [MODULES OVER BIALGEBRA!]

$\boxed{\text{NOTATION}}$

$(\mathcal{M}, \underbrace{\otimes_{T_\circledast(A)}}_{\otimes_{\mathcal{M}}}, \underbrace{T_\circledast(A)}_{I_\otimes})$ IS SYMM. MON CAT [MODULES OVER COMM. ALG!]

$\boxed{\text{NOTATION}}$

$\boxed{\text{THM}}$ $(\mathcal{M}, \otimes_{\mathcal{M}}, I_\otimes, \boxtimes_{\mathcal{M}}, I_\boxtimes, sh)$ IS A 2-MON CAT.

SOME INTERCHANGE LAW

$\boxed{\text{FOR QFT: } B = T_*(S_{C^\infty(M)}(I^! E))}$ BIALGEBRA (IN $\mathcal{M}$!)

(DOMAIN OF OUR LAPLACE PAIRING)

§6. MORE PARTIAL ALGEBRAS

$\boxed{\text{THM}}$ WE HAVE

- * PARTIAL ALG STRUCTURE ON $\mathcal{D}'(M)$ OVER $C^\infty(M)$ $\rightsquigarrow$ PARTIAL ALG STRUCTURE ON $T_* \mathcal{D}'(M)$ IN $(\mathcal{M}, \otimes_{\mathcal{M}}, I_\otimes)$
- * EXTERNAL PRODUCT OF $\mathcal{D}'(M)$ $\rightsquigarrow$ ALG STRUCTURE ON $T_* \mathcal{D}'(M)$ IN $\mathrm{Fun}_C(\mathcal{F}, \mathcal{C})$

WE CAN COMBINE THEM TO GET $\boxed{\text{TWO}}$ PARTIAL ALG! STRUCTURES ON

$\mathcal{D}(M) := T_* \mathcal{D}'(M) \otimes_C T_* \mathcal{D}'(M)$

[LEFT / RIGHT]

$\boxed{\text{THM}}$ THE PROPAGATOR Δ CAN BE EXTENDED UNIQUELY TO A [LEFT/RIGHT] LAPLACE PAIRING FROM $\mathcal{B} = T_*(S_{C^\infty(M)}\Gamma' J' E)$ TO $\mathfrak{D}\mathcal{A}(M)$ IN THE 2-MON CAT $\mathcal{M}$.

$\boxed{\text{SOME ASSUMPTIONS ON THE WF OF } \Delta \text{ \& } \delta}$