On the category of BV-theories
(quantum L_∞-algebras)

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POTSDAM 2022

work in progress

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V - dg v.s. (degree-wise finite-dim)

ω - odd symplectic form

$|\omega| = -1$

Q - differential, $\omega(Q\otimes 1 + Q\otimes 1) = 0$

Solutions to QME $\Longleftrightarrow$ quantum L_∞-algebras

$$S = \sum \frac{\hbar^g}{n!} S_n^g$$

$$S_n^g = f_n^g(e_{i_1}, \dots, e_{i_n}) \phi^{i_1} \dots \phi^{i_n} \in \mathrm{Fun}(V)$$ (formal power series in $\hbar$)

$f_n^g : S^n(V) \to k$ $\quad |f_n^g| = 0$

e_i - homogeneous basis of V $\qquad \omega(e_i, e_j) = \omega_{ij}$ with ω^{ij} inverse

ϕ^i - dual basis of V^*

$$\Delta = \pm \omega^{ij} \frac{\partial}{\partial \phi^i} \frac{\partial}{\partial \phi^j} \quad \text{on } \mathrm{Fun}(V)$$

$$\{f, g\} = \pm \omega^{ij} \frac{\partial f}{\partial \phi^i} \frac{\partial g}{\partial \phi^j},$$

$f_2^0 = \omega(\cdot, Q\cdot)$

$S = S_{tree} + S_I$

$$\Delta e^{\frac{S}{\hbar}} = 0 \iff \hbar \Delta S + \frac{1}{2}\{S, S\} = 0 \iff (\hbar\Delta + \{S, \cdot\})^2 = 0$$

QME

$$\Updownarrow$$

q-L_∞-algebra

$$\sum_{\substack{n_1+n_2=n \\ \sigma \in Sh(n_1, n_2)}} \sum_{g_1+g_2=g} \pm \omega^{ij} f^{g_1}_{n_1+1}(x_{\sigma(1)}, \dots, x_{\sigma(n_1)}, e_i) \cdot f^{g_2}_{n_2+1}(x_{\sigma(n_1+1)}, \dots x_{\sigma(n_1+n_2)}, e_j)$$

$$\pm \omega^{ij} f^{g-1}_{n+2}(x_1, \dots, x_n, e_i, e_j) = 0$$

$g = 0$ cyclic L_∞

- ω - nondeg.

$$\omega(x_1, m^g_{n-1}(x_2, \dots, x_n)) := f^g_n(x_1, \dots, x_n)$$

$m^g_{n-1} : S^{n-1}(V) \longrightarrow V \qquad |m^g_{n-1}| = 1$

Equivalently algebraic relations from m^g_n's.

From now on no explicit $\hbar$.

BV-integration $\Longleftrightarrow$ homological perturbation lemma

S-matrix $\longleftrightarrow$ minimal model (decomposition theorem) for q-L_∞-algebra

· Hodge decomposition

$$V = H \oplus C \oplus B$$

$$\begin{tikzcd} C \arrow[r, bend left, "Q"] & B \arrow[l, bend left, "k"] \end{tikzcd}$$

H - cohomology of Q

k - partial inverse to Q

$\omega(k \otimes 1 + 1 \otimes k) = 0$

correspondingly for Δ, $\{ , \}$

$$\omega = \omega' + \omega'' =: \omega|_H \oplus \omega|_{C \oplus B},$$

C, B - Lagrangian in $C \oplus B$

$$k \circlearrowright (V, Q) \underset{i}{\overset{p}{\rightleftarrows}} (H, 0)$$

$pQ = 0 \quad , \quad Qi = 0$

$|k| = -1 \qquad k^2 = 0 \quad ki = 0 \quad pk = 0$

$Qk + kQ = ip - 1$

special deformation retract (SDR)

Tensor trick $\Rightarrow$ SDR on $\mathcal{F}(V)$

$$k \circlearrowright (\mathcal{F}(V), Q) \underset{i}{\overset{p}{\rightleftarrows}} (\mathcal{F}(H), 0)$$

k, p, i - induced from V

- Perturbation of Q by δ $\Rightarrow$ new SDR

$$\begin{tikzcd} k_1 \circlearrowright \left(\mathcal{F}(V), Q+\Delta \right) \arrow[r, shift left, "p_1"] & \left(\mathcal{F}(H), 0+\Delta_H \right) \arrow[l, shift left, "i_1"] \end{tikzcd}$$

$\Delta_H = \Delta|_H$

$$p_1 = p(1-\Delta k)^{-1}$$
$$i_1 = (1-k\Delta)^{-1}$$
$$k_1 = k(1-\Delta k)^{-1}$$

in particular
$$p_1(\Delta+Q) = \Delta_H p_1$$

Hence $(\Delta+Q)e^{S_I} = 0 \Rightarrow \Delta_H e^{W} = 0$

S_I-interaction

$$\underline{e^{W} = p(1-\Delta k)^{-1} e^{S}}$$

W - effective action on H (S-matrix)
- minimal model for correspond. q-L_∞-alg.

$S \mapsto W$ quasi-isomorphism (iso on cohomology)

- Decomposition theorem from HPL

$$(\Delta+Q)k_1 + k_1(\Delta+Q) = i_1 p_1 - 1 \quad / e^S$$

$$(\Delta+Q)k_1 e^S = e^{i(W)} - e^S$$

from $(\Delta+Q)$-exactness of $e^{i(W)} - e^S$, it follows that there is a canonical map $\phi : U \to U$
(non-linear, diffeo starting with 1)

$$\phi^* \omega = \omega$$

$$\phi^* \left(e^{S_{tree} + S_I} d^{\frac{1}{2}} V \right) = e^{S_{tree} + i(W)} d^{\frac{1}{2}} V$$

Theorem $\quad e^{W} = \int_C e^{S} \Big|_{H \oplus C}$

gauge fixing

<u>Note</u> $\quad e^{W}$ comes from $P \circ \phi$, relating ω and ω_H

i.e. a Poisson map

What is a category of BV-theories (q-L_∞-algebras) such that minimal model and decomposition theorem will correspond to some morphism in this category?

From above it is natural consider Poisson maps - too restrictive, Poisson maps go only in one direction w.r.t. dimension

Can we do better?

For minimal model: Consider half-density defined by action S, $e^S d^{\frac{1}{2}}V \in Dens^{\frac{1}{2}}(V)$

and a Dirac's δ_L density supported on $L = C \times \mathrm{diag}\, H$, $\delta_L \in Dens^{\frac{1}{2}}(\bar{V} \times H)$

Hence $\delta_L e^S d^{\frac{1}{2}}V \in Dens^{\frac{1}{2}}(\bar{V} \underset{12}{\times} H) \otimes Den^{\frac{1}{2}}(V)$

$$Dens(V) \otimes Dens^{\frac{1}{2}}(H)$$

So

$$\int \delta_L e^S d^{\frac{1}{2}}V \in Dens^{\frac{1}{2}}(H)$$

Observation $\displaystyle\int_V \delta_{C \times \mathrm{diag}\, H}\, e^{S} d^{\frac{1}{2}}V = e^{W} d^{\frac{1}{2}} H$

For decomposition theorem:

Consider Lagrangian submanifold $L = \mathrm{graph}\,\phi \subset \bar{V} \times V$

and $\delta_{\mathrm{graph}\,\phi} \in \mathrm{Dens}^{\frac{1}{2}}(\bar{V} \times V)$

Observation $\displaystyle\int \delta_{\mathrm{graph}\,\phi}\, e^{S} d^{\frac{1}{2}} V = \phi^{*}(e^{S} d^{\frac{1}{2}} V)$

$= e^{S_{\mathrm{tree}} + iW} d^{\frac{1}{2}} W$

Remark In both cases are all $\frac{1}{2}$-densities

Δ-closed

$$\left(\Delta\left(f\, d^{\frac{1}{2}}V\right) := \Delta f \cdot d^{\frac{1}{2}}V \right)$$

<u>Definition</u> (Quantum Odd Symplectic Category, Ševera)
QOSC

- Objects odd symplectic (super)manifolds
- Morphisms $M_1 \to M_2$ are generalized $\frac{1}{2}$-densities

$$\mathrm{Dens}^{\frac{1}{2}}(\bar{M}_1 \times M_2)$$

- Composition

$$\mathrm{Dens}^{\frac{1}{2}}(\bar{M}_1 \times M_2) \otimes \mathrm{Dens}^{\frac{1}{2}}(\bar{M}_2 \times M_3) \xrightarrow{\int_{M_2}} \mathrm{Dens}^{\frac{1}{2}}(\bar{M}_1 \times M_3)$$

Further, $\Delta : \mathrm{Dens}^{\frac{1}{2}}(M) \longrightarrow \mathrm{Dens}^{\frac{1}{2}}(M)$ (Khudaverdian)

Proposition In QOSC, composition of Δ-closed morphism is Δ-closed

Prelim Def For M an odd symplectic manifold a solution to QME is a Δ-closed element of $\mathrm{Dens}^{\frac{1}{2}}(M)$.

$(M, F) \longrightarrow (N, G)$ morphism is defined as a Δ-closed $\rho \in \mathrm{Dens}^{\frac{1}{2}}(\bar{M} \times N)$ such that

$$\int_M F \rho = G.$$

Examples
- minimal model
- decomposition theorem

- Definition is asymmetric between M and N, a better one is under construction, details hopefully soon
- The more symmetric definition also indicates the right notion of T-duality of BV-theories (q-L_∞-algebras)

Outline

Consider linear QOSC

$L \subset \bar{M} \times N$ — all linear spaces

Then L is necessarily of the form

$$L = I_M \times \text{graph}\,\phi \times I_N$$

I_M, I_N isotropic in M and N resp.

$$\phi : T_M := \frac{I_M^\perp}{I_M} \longrightarrow T_N := \frac{I_N^\perp}{I_N} \quad \text{linear iso}$$

(symplectic reductions)

Consider

$$\int_M e^{S_M} \delta_L \quad d^{\frac{1}{2}}M$$

It is easy to see that we get $\frac{1}{2}$-density on N of the form

$$e^{S_{T_N}} \, d^{\frac{1}{2}}N$$

S_{T_N} does depend only on T_N

Similarly $M \longleftrightarrow N$, the same L

In this case it seem more reasonalee sk that

$$e^{S_{T_M}} d^{\frac{1}{2}} T_M = \phi^* \left(e^{S_{T_N}} d^{\frac{1}{2}} T_N \right)$$

for a morphism

$$(M, S_M) \simeq (N, S_N)$$

T-duality ?

$(M, S_M) \overset{T}{\simeq} (\tilde{M}, S_{\tilde{M}})$ if $\exists\, (N, S_N)$ and $\tilde{}$ two symplectic reductions corresp. to $l_N, \tilde{l}_N$ resp

$L = \{(n, \Pi(n)),\ n \in l_N^{\perp}\}$

$$\begin{tikzcd} & N \arrow[dl, "\Downarrow"'] \arrow[dr, Rightarrow] & \\ M & & \tilde{M} \end{tikzcd}$$

$\tilde{L} = \dots$

s.t.

$$e^{S_M} d^{\frac{1}{2}} M = \int e^{S_N} \rho_L\, d^{\frac{1}{2}} N \quad \text{and} \quad e^{\tilde{S}_M} d^{\frac{1}{2}} \tilde{M} = \int e^{S_N} \tilde{\rho}_L\, d^{\frac{1}{2}} N$$

$$\begin{tikzcd}
 & N \arrow[dl] \arrow[dl, dashed, bend right, "\wr"'] \arrow[dr, "\sim"] & \\
M \arrow[rr, "L"'] \arrow[rr, bend left=60, "\alpha"] \arrow[dr] & & \tilde{M} \arrow[dl] \\
 & M \cap N &
\end{tikzcd}$$

$S_M \overset{T}{\simeq} S_{\tilde{M}}$ doesn't necessarily imply $S_M \simeq S_{\tilde{M}}$

- Remark – easy to interpret using HPL

$$M = H \oplus B \oplus C \qquad I_M \subset C \simeq M/\mathrm{Ker}\, Q$$