Monotone Cumulants and Schröder Trees.

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In classical probability, if X and Y are independent random variables then
$$\mathbb{E}(X^m Y^n) = \mathbb{E}(X^m)\mathbb{E}(Y^n), \quad \forall\, m,n \geq 0.$$

Algebraically, independence is a "recipe" to compute mixed moments.

Def. A non-commutative probability space is a pair $(\mathcal{A}, \varphi)$ where $\mathcal{A}$ is a complex unital algebra and $\varphi: \mathcal{A} \to \mathbb{C}$ is a linear functional such that $\varphi(1_{\mathcal{A}}) = 1$.

Example i) Let $L^{-\infty} := L^{\infty}(\Omega, \mathbb{P})$ be the algebra of complex random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with finite moments of all orders, and $\mathbb{E} = \int_\Omega d\mathbb{P}$. Then $(L^{\infty-}, \mathbb{E})$ is a non-comm. prob. space.

ii) $M_d(\mathbb{C})$ algebra of $d \times d$ complex matrices and $Tr: M_d(\mathbb{C}) \to \mathbb{C}$ trace. Then $(M_d(\mathbb{C}), \frac{1}{d}Tr)$ is a non-comm. prob. space.

Elements $a \in \mathcal{A}$ are called random variables, and $\varphi(a)$ is called the moment of a.

It can be shown that there are only five notions of non-comm. Independence.

Def. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and $\mathcal{A}_1, \dots, \mathcal{A}_N \subset \mathcal{A}$ subalgebras. Take any elements $a_j \in \mathcal{A}_{i_j}$, for $1 \leq j \leq n$, where $i_1, \dots, i_n \in \{1, \dots, N\}$ are such that $i_1 \neq i_2$, $i_2 \neq i_3, \dots, i_{n-1} \neq i_n$.

i) Assume that every $\mathcal{A}_i$ is unital. We say that $\{\mathcal{A}_i\}_{i=1}^N$ are freely independent if $\varphi(a_1 \cdots a_n) = 0$ when $\varphi(a_j) = 0$, $\forall\, 1 \leq j \leq n$.

ii) We say that $\{\mathcal{A}_i\}_{i=1}^N$ are Boolean independent if $\varphi(a_1 \cdots a_n) = \varphi(a_1) \cdots \varphi(a_n)$.

iii) We say that $\{\mathcal{A}_i\}_{i=1}^N$ are monotone independent if
$$\varphi(a_1 \cdots a_n) = \varphi(a_l)\,\varphi(a_1 \cdots a_{l-1} a_{l+1} \cdots a_n)$$
if $i_{l-1} < i_l$ and $i_l > i_{l+1}$.

Example Take $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ free unital subalgebras, and $a \in \mathcal{A}_1$, $b \in \mathcal{A}_2$. We compute $\varphi(ab)$. Since $a - \varphi(a)1_{\mathcal{A}} \in \mathcal{A}_1$, $b - \varphi(b)1_{\mathcal{A}} \in \mathcal{A}_2$, and $\varphi(a - \varphi(a)1_{\mathcal{A}}) = 0 = \varphi(b - \varphi(b)1_{\mathcal{A}})$, we have
$$0 = \varphi\big((a - \varphi(a)1_{\mathcal{A}})(b - \varphi(b)1_{\mathcal{A}})\big) = \varphi(ab) - \varphi(a)\varphi(b) - \varphi(b)\varphi(a) + \varphi(a)\varphi(b)\varphi(1_{\mathcal{A}})$$
$$= \varphi(ab) - \varphi(a)\varphi(b)$$

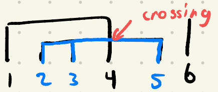
Thus $\varphi(ab) = \varphi(a)\varphi(b)$.

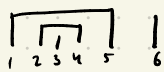
For each of the above notions of independence, it is possible to construct a theory of non-commutative probability: convolutions, central limit theorems, Lévy processes.

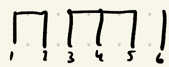
In particular, we have non-comm. notions of cumulants.

Def. A partition of $[n] := \{1, \ldots, n\}$ is a collection of subsets of $[n]$ that are non-empty, pairwise disjoint, and whose union is $[n]$.

A <u>non-crossing partition</u> of $[n]$ is a partition π such that there are no elements $a < b < c < d$ and different blocks $V, W \in \pi$ such that $a, c \in V$ and $b, d \in W$.







Crossing partition
$\pi = \{\{1,4\}, \{2,3,5\}, \{6\}\}$

Non-crossing partition
$\{\{1,5\}, \{2,4\}, \{3\}, \{6\}\}$

Interval partition
$\{\{1,2\}, \{3,4,5\}, \{6\}\}$.

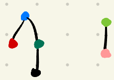
An <u>interval partition</u> of $[n]$ is a non-crossing partition whose blocks are interval.

$NC(n)$ is the set of non-crossing partitions on $[n]$.
$Int(n)$ is the set of interval partitions on $[n]$.

- $NC(n)$ is a poset: $\pi \leq \sigma$ in $NC(n)$ if the blocks of π are contained in the blocks of σ.

$1_n = \{\{1, \ldots, n\}\}$ is the maximal element, $0_n := \{\{1\}, \{2\}, \ldots, \{n\}\}$ is minimal.

- Every $\pi \in NC(n)$ is a poset: For $V, W \in \pi$, we say $V \leq W$ if W is nested in V. The forest of nesting of π is a forest of planar rooted trees that encodes the poset structure of π.

$\pi =$  , $t(\pi) =$ 

- For any tree t such that is the grafting of the subtrees $s_1, \ldots, s_m$, we define the tree factorial by

$$t! = |t|\, s_1! \cdots s_m! \,, \quad \text{with} \quad \bullet ! = 1 \,,$$ and $\bullet$ is the single-vertex tree.

If $f = t_1 \cdots t_n$ is a forest, then $f! = t_1! \cdots t_n!$

Combinatorial definition of cumulants.

Def. Let $(\mathcal{A}, e)$ be a ncps.

i) The <u>free cumulants</u> form the family of multilinear functionals $\{k_n : \mathcal{A}^n \longrightarrow \mathbb{C}\}_{n \geq 1}$ recursively defined by

$$e(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \prod_{V \in \pi} k_{|V|}(a_1, \ldots, a_n | V)$$

where if $V = \{i_1 < i_2 < \cdots < i_s\}$ then we write

$K_{|V|}(a_1, \ldots, a_n | V) = K_s(a_{i_1}, a_{i_2}, \ldots, a_{i_s})$. In general, for a family $\{f_n : A^n \to \mathbb{C}\}_{n \geq 1}$ and $\pi \in NC(n)$, we write $f_\pi(a_1, \ldots, a_n) = \prod_{V \in \pi} f_{|V|}(a_1, \ldots, a_n | V)$.

ii) The Boolean cumulants form the family of multilinear functionals $\{b_n : A^n \to \mathbb{C}\}_{n \geq 1}$ recursively defined by

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in Int(n)} b_\pi(a_1, \ldots, a_n).$$

iii) The monotone cumulants form the family of multilinear functionals $\{h_n : A^n \to \mathbb{C}\}_{n \geq 1}$ recursively defined by

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \frac{1}{t(\pi)!} h_\pi(a_1, \ldots, a_n).$$

- Since $NC(n)$ and $Int(n)$ are posets, by Möbius inversion we can write

$$K_n(a_1, \ldots, a_n) = \sum_{\pi \in NC(n)} \text{Möb}(\pi, 1_n)\, \varphi_\pi(a_1, \ldots, a_n)$$

$$b_n(a_1, \ldots, a_n) = \sum_{\pi \in Int(n)} (-1)^{|\pi| - 1} \varphi_\pi(a_1, \ldots, a_n),$$

$\forall n \geq 1$, $a_1, \ldots, a_n \in A$, where $\varphi_n(a_1, \ldots, a_n) := \varphi(a_1 \cdots a_n)$.

However, Möbius inversion cannot be applied to the monotone case.

Question: Describe the coefficients $\{\alpha(\pi)\}_{\pi \in \bigcup_{n \geq 1} NC(n)}$ such that

$$h_n(a_1, \ldots, a_n) = \sum_{\pi \in NC(n)} \alpha(\pi)\, \varphi_\pi(a_1, \ldots, a_n)$$

$\forall n \geq 1$ and $a_1, \ldots, a_n \in A$.

Idea: Hopf-algebraic approach for non-commutative probability.

Double tensor unshuffle bialgebra

(Ebrahimi-Fard and Patras). Let (A, φ) be a ncps. Define

$$T_+(A) = \bigoplus_{n \geq 1} A^{\otimes n}, \qquad T(T_+(A)) = \bigoplus_{n \geq 0} T_+(A)^{\otimes n}$$

$T(T_+(A))$ is a Hopf algebra, with concatenation as product. Tensors in the double tensor algebra are denoted by bars:

$$w_1 | w_2 | \cdots | w_k \in T_+(A)^{\otimes k} \quad \text{if} \quad w_1, \ldots, w_k \in T_+(A).$$

Coproduct is given as follows: if $w = a_1 \cdots a_n \in A^{\otimes n} \subset T_+(A)$, then

$$\Delta(a_1 \cdots a_n) = \sum_{S \subseteq [n]} a_S \otimes a_{J_1} | \cdots | a_{J_r}$$

where if $S = \{i_1 < \cdots < i_\ell\}$ then $a_S = a_{i_1} \cdots a_{i_\ell}$, and $J_1, \ldots, J_r$ are the connected components of $[n] \setminus S$.

The main property of Δ is that it can be split into $\Delta = \Delta_< + \Delta_>$ by

$$\Delta_<(a_1 \cdots a_n) = \sum_{1 \in S \subseteq [n]} a_S \otimes a_{J_1} | \cdots | a_{J_r}, \qquad \Delta_>(a_1 \cdots a_n) = \sum_{1 \notin S \subseteq [n]} a_S \otimes a_{J_1} | \cdots | a_{J_r}$$

Now, for $f, g \in \mathrm{Lin}(T(T_+(A)), \mathbb{C})$, we define

$$f * g = m_{\mathbb{C}} \circ (f \otimes g) \circ \Delta$$ convolution product, associative

$$f < g = m_{\mathbb{C}} \circ (f \otimes g) \circ \Delta_<$$
$$f > g = m_{\mathbb{C}} \circ (f \otimes g) \circ \Delta_>$$
} non-associative products

$(\mathrm{Lin}(T(T_+(A)), \mathbb{C}, <, >)$ is a unital shuffle algebra.

- G group of characters on $T(T_+(A))$
  $\mathfrak{g}$ Lie algebra of infinitesimal characters on $T(T_+(A))$.

By general theory, the map $\exp^*: \mathfrak{g} \to G$ given by $\exp^*(\alpha) = \sum_{n \geq 0} \frac{1}{n!} \alpha^{*n}$ is a bijection. Moreover, $\mathcal{E}_<(\alpha) = \sum_{n \geq 0} \alpha^{<n}$, $\alpha^{<n} = \alpha < \alpha^{<n-1}$, $\alpha^{<0} = \underset{\text{counit}}{\varepsilon}$ and $\mathcal{E}_>(\alpha) = \sum_{n \geq 0} \alpha^{>n}$, $\alpha^{>n} = \alpha^{>n-1} > \alpha$, are both bijections $\mathfrak{g} \to G$.

Link with non-commutative probability

Let (A, φ) be a ncps, and consider $T(T_+(A))$. Define a character

$\Phi: T(T_+(A)) \to \mathbb{C}$ by $\Phi(w) = \varphi(a_1 \cdots a_n)$ if $w = a_1 \cdots a_n \in A^{\otimes n}$.

Thm (EF, P). The infinitesimal characters $\kappa, \beta, \rho \in \mathfrak{g}$ such that

(1) $\Phi = \mathcal{E}_<(\kappa) = \mathcal{E}_>(\beta) = \exp^*(\rho)$ satisfy that

$$\kappa(w) = k_n(a_1, \ldots, a_n), \qquad \beta(w) = b_n(a_1, \ldots, a_n), \qquad \rho(w) = h_n(a_1, \ldots, a_n)$$

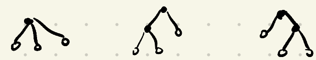
for any $w = a_1 \cdots a_n \in A^{\otimes n}$.

Indeed, when we evaluate (1) on a word $w = a_1 \cdots a_n \in A^{\otimes n}$, we obtain the moment-cumulant relations. A first approach to solve the question is evaluate $\rho = \log^* \Phi$, but it is not clear.

A Hopf algebra of Schröder trees

Def A Schröder tree is a planar rooted tree such that each of its internal vertices has at least two children.

Example Schröder trees with three leaves

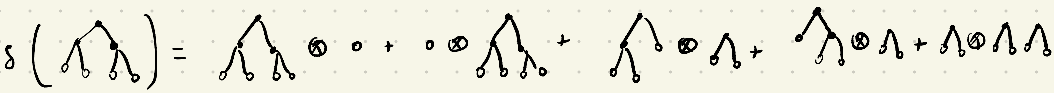


Denote $ST(n) := \{t : t \text{ Schröder tree with } n+1 \text{ leaves}\}$, $ST = \bigcup_{n \geq 0} ST(n)$

Consider the non-commutative polynomial algebra $H_S := \mathbb{C}\langle t : t \in ST \rangle / (\circ - 1)$

H_S is also a coalgebra, with coproduct given by a non-commutative version of the Connes-Kreimer coproduct, and where only internal vertices are considered.

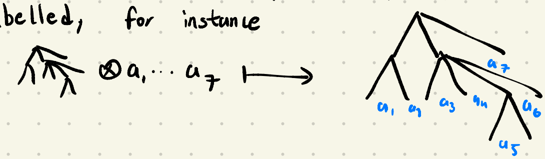
Example



H_S is a connected graded Hopf algebra, with $\deg(t) = n$ if $t \in ST(n)$.

Decorated version. For a vector space A, let $H_S(A) := \bigoplus_{n \geq 0} (H_S(n) \otimes A^{\otimes n})$.

An element $t \otimes w$ is identified with a Schröder tree whose sectors between leaves are labelled, for instance



$H_S(A)$ is a connected graded Hopf algebra.

Proposition (Josuat-Vergès et al) If $i : T(T_+(A)) \to H_S(A)$ is the algebra morphism given by

$$i(a_1 \cdots a_n) = \sum_{t \in ST(n)} t \otimes a_1 \cdots a_n$$

then i is a coalgebra (and unshuffle bialgebra) morphism.

Returning to non-commutative probability

Given a ncps (A, φ), construct the character $\tilde{\Phi} : H_S(A) \to \mathbb{C}$ by

$$\tilde{\Phi}(t \otimes a_1 \cdots a_n) = \begin{cases} \varphi(a_1 \cdots a_n) & \text{if } t \text{ is a corolla with } n+1 \text{ leaves} \\ 0 & \text{otherwise.} \end{cases}$$

- It is clear that $\Phi = \tilde{\Phi} \circ i$.

Now consider the infinitesimal character $\tilde{\rho} : H_S(A) \to \mathbb{C}$ s.t. $\tilde{\Phi} = \exp^*(\tilde{\rho})$.

Proposition If $\rho = \tilde{\rho} \circ i$, then $\Phi = \exp^*(\rho)$, i.e. ρ is associated to monotone cumulants.

Idea. Compute $\tilde{\rho} = \log^{*}(\tilde{\Phi}) = \sum_{k\geq 1} \frac{(-1)^{k+1}}{k} \tilde{\Phi}_0^{*k}$, with $\tilde{\Phi}_0 = \tilde{\Phi} - \varepsilon$,
counit on $H_S(A)$.

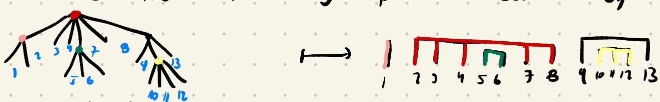
- By Prop. $\rho(a_1\cdots a_n) = \sum_{t\in ST(n)} \tilde{\rho}(t\otimes a_1\cdots a_n)$.

Also $\tilde{\Phi}_0^{*k} = m_C^{[k]} \circ \tilde{\Phi}_0^{\otimes k} \circ \delta^{[k]}$, with $\delta^{[k]}(t\otimes a_1\cdots a_n) = \sum_{\substack{\text{certain iterated}\\ \text{admissible cuts}}} f_1\otimes\cdots\otimes f_k$

The definition of $\tilde{\Phi}_0$ allows us to identify the terms in $\delta^{[k]}$, that will produce a non-zero contribution.

Proposition $\tilde{\Phi}_0^{*k}(t\otimes a_1\cdots a_n) = w_k(sk(t))\, \varphi_{\pi(t)}(a_1,\dots,a_n)$, $\forall\, a_1,\dots,a_n\in A$, $t\in ST(n)$,
where:

$sk(t)$ is the subtree of t generated by its internal vertices;
$w_k(sk(t))$ is the number of surjective order-preserving functions $f: sk(t)\to [k]$;
$\pi(t)$ is the non-crossing partition defined by the rule



If $w(sk(t)) = \sum_{k=1}^{|sk(t)|} \frac{(-1)^{k+1}}{k} w_k(sk(t))$ as in Frédéric's talk, we obtain

$$\tilde{\rho}(t\otimes a_1\cdots a_n) = w(sk(t))\, \varphi_{\pi(t)}(a_1,\dots,a_n).$$

Hence $\rho(a_1\cdots a_n) = \sum_{t\in ST(n)} w(sk(t))\, \varphi_{\pi(t)}(a_1,\dots,a_n)$, $\forall\, a_1,\dots,a_n\in A$.

Grouping terms: $h_n(a_1,\dots,a_n) = \sum_{\pi\in NC(n)} \alpha(\pi)\, \varphi_\pi(a_1,\dots,a_n)$,

where $\boxed{\alpha(\pi) = \sum_{\substack{t\in ST(n)\\ \pi(t)=\pi}} w(sk(t)), \quad \forall\, \pi\in NC(n).}$ ∎